AFWAL-TR-87-3097

AN INTEGRAL EQUATION FOR THE LINEARIZED SUPERSONIC FLOW OVER A WING

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February 1988

Interim Report for the Period November 1986 to August 1987

FLIGHT DYNAMICS LABORATORY
AIR FORCE WRIGHT AERONAUTICAL LABORATORIES
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NOTICE

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REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution is unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) UDR-TR-87-91	5. MONITORING ORGANIZATION REPORT NUMBER(S) AFWAL-TR-87-3097
6a. NAME OF PERFORMING ORGANIZATION University of Dayton Research Institute 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION Flight Dynamics Laboratory (AFWAL/FIBRC) AF Wright Aeronautical Laboratories
6c. ADDRESS (City, State and ZIP Code) 300 College Park Dayton, OH 45469	7b. ADDRESS (City, State and ZIP Code) Wright-Patterson Air Force Base Dayton, OH 45433-6553
8a. NAME OF FUNDING/SPONSORING ORGANIZATION 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER F33615-86-C-3200
8c. ADDRESS (City, State and ZIP Code)	10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
61102F	2304	N1	22

11. TITLE (Include Security Classification) An Integral Equation for the Linearized Supersonic Flow Over a Wing

12. PERSONAL AUTHOR(S)
Karl G. Guderley

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Interim	FROM Nov 86 TO Aug 87	February 1988	105

16. SUPPLEMENTARY NOTATION

17. COSATI CODES		
FIELD	GROUP	SUB. GR.
01	01	
20	04	

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
Supersonic Flows, Integral Equation, Conical Flows, Lorentz Transformation

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

In the first formulation, the integral equation for linearized steady supersonic flows appears in a form where it is necessary to approach the planform from above or below by a limiting process. In the present report, the problem is transformed analytically in such a manner that this limiting process no longer appears and that the resulting expressions are numerically tractable. It is believed that such a formulation gives more freedom to take the particularities of a given problem into account. The formulation is applied to compute the conical field which arises at the top of an airfoil. A possible numerical approach to the solution of the integral equation in the general case is described, but only in a rough outline.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE NUMBER (Include Area Code)	22c. OFFICE SYMBOL
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FOREWORD

This report was written under Contract F33615-86-C-3200 entitled "A Study of Linearized Integral Equations for Steady and Oscillatory Supersonic Flow," to the University of Dayton for the Aeroelastic Group, Analysis and Optimization Branch, Structures and Dynamics Division (AFWAL/FIBRC), Air Force Wright Aeronautical Laboratories, Wright-Patterson Air Force Base, Ohio. The work was conducted under Program Element No. 61102F, Project No. 2304, Task N1, and Work Unit 22.

The work was performed during the period of November 1986 through August 1987. Dr. Karl G. Guderley of the University of Dayton Research Institute was Principal Investigator, Dr. Charles L. Keller, AFWAL/FIBRC, (513) 255-7384, was Program Manager.

The author would like to express his appreciation for the excellent typing of Ms. Kathleen Reineke.

TABLE OF CONTENTS

APPENDICES

LIST OF ILLUSTRATIONS

SECTION I

INTRODUCTION

This report derives an integral equation for the linearized steady supersonic potential flow over a wing. As in the case of a subsonic flow a difficulty arises because of the singularity of the kernel of the integral equation. The singularity makes it necessary to express the upwash at the wing by a limiting process in which one approaches the wing from the interior of the flow field. Each integral equation formulation is based on the use of a fundamental solution. The idea of using such fundamental solutions for the representation of the flow field is quite old. It appears for instance in so called "box methods". These methods start immediately with a discretization of the problem and in this manner reduce the difficulties caused by the singularity of the kernel. In contrast the present approach uses analytical means to transform the initial formulation so that in the final formulation one deals with tractable quantities and the limiting process in which one approaches the planform no longer appears explicitly. A discretization must, of course, again be carried out, but it appears only after the analytical preparation has been carried out. The author believes that this procedure gives greater flexibility in taking the particularities of a specific problem into account. Of course, one may then arrive at a numerical procedure that is less automatic.

The fundamental solution for supersonic flows is identical with that for subsonic flows, except that certain terms change their signs. If (x,y,z) is a point of the flow field and (ξ,η) a point of the planform then the fundamental solution is given by $1/r$, where $r = [(x-\xi)^2 + (1-M^2)(\eta-y)^2 + (1-M^2)z^2]^{1/2}$. In the present context, x is greater than ξ; the author has therefore consistently written $(x-\xi)^2$ instead of $(\xi-x)^2$)

In subsonic flow the coefficient $(1-M^2)$ is positive, in supersonic flow it is negative. This has a profound effect. In subsonic flows all points of the wing and of the wake have an

effect on the potential in the field, in supersonic flow only points of the wing and wake have an influence that lie within the forecone of the point (x,y,z), that is points (ξ,η) for which

$$[(x-\xi)^2 - (M^2-1)(\eta-y)^2-(M^2-1)z^2] > 0 \text{ and } x-\xi > 0$$

This is a significant simplification. A singularity is encountered if this expression is zero. Roughly speaking one expresses the potential by the integral over ξ and η of such fundamental solutions multiplied by a weight function that depends upon ξ and η. In a subsonic flow, if one evaluates the potential (or rather its derivatives) at a point (x,y,z) then $r \neq 0$ for every point of the flow field for which $z \neq 0$. If the point (x,y,z) lies on the planform (in the $\xi\eta$-plane, i.e., for $z = 0$), one has a singularity of the fundamental solution at $\xi = x$, $\eta = y$. This makes the limiting process $z \to 0$ necessary.

In supersonic flow the expression r vanishes along the entire surface of the cone $r > 0$ with tip at point x,y,z. For $z \neq 0$ the evaluation of the potential or its derivatives must take into account the contribution of the fundamental solutions, with values of ξ,η of the planform, for which $r > 0$. These values lie within the hyperbola in the $\xi\eta$-plane given by

$$(x-\xi)^2 - (M^2-1)(\eta-y)^2 = (M^2-1)z^2, \quad x-\xi \geq 0.$$

At this hyperbola one has $r = 0$, the integrand of the integral equation is singular along this curve, even for $z \neq 0$. The singularity becomes more pronounced if one takes derivatives. This is remedied by a single transformation, but then the image of the leading edges under this transformation will be z-dependent. The effect is minor because for the problem treated here the weight function for the fundamental solutions is zero at the leading edge.

In Section II of this report, the approaches to the problem by the velocity and by the acceleration potential are compared with each other. For supersonic flow, at least, the author favors

the velocity potential, because the pertinent fundamental solution does not introduce a singularity at the wake of the points (ξ,η) of the planform for which it is defined. Such singularities make the upwash field less smooth, when one discretizes the integral equation. This generates an uncertainty, if one imposes upwash conditions at individual control points. A main advantage of the acceleration potential lies in the fact that it anticipates the contribution of the wake. But in most supersonic flows the wake has no influence on the pressure distribution over the air foil.

In Section III the basic equations are compiled. This includes the introduction of the Lorentz transform which will prove useful in the treatment of conical fields. Section IV describes the problem in general terms. In Section V the analysis of the general case is prepared by a two-dimensional example. The general integral equation is derived in Section VI. This discussion deals mainly with the leading concepts. Some detailed investigations which, although desirable for mathematical completeness, only confirm results that can be expected, are found in Appendix A. Section VII applies the integral equation found in Section VI to conical fields, they occur in the vicinity of the tip of the airfoil. Here the Lorentz transformation is applied to suppress an unpleasant numerical singularity. In Section VIII some remarks about a possible discretization are made. The treatment of contribution of points (ξ,η) at a distance from the point (x,y) for which the upwash is evaluated is rather sketchy; more details are given for the contributions of points (ξ,η) in the immediate vicinity of the point (x,y).

Mathematical details which would have been too disruptive in the presentations of the main ideas are found in a number of appendices.

SECTION II

COMPARISON BETWEEN THE VELOCITY AND THE ACCELERATION POTENTIAL

As is well-known, the analytical formulations for the linearized flow over a wing in terms of the velocity potential and in terms of the acceleration potential are based on the same simplifications and are, therefore, equivalent. Differences arise, however, in the numerical realization. Here are the main points.

1. The velocity potential arises from the acceleration potential by an integration along the stream lines (of the undisturbed flow). A piecewise linear approximation for the velocity potential, therefore, gives a piecewise constant function for the acceleration potential and with it, at least in steady flows, piecewise constant pressures. To obtain approximations of the same quality one, therefore, needs smoother approximating functions for the velocity potential than for the acceleration potential.

2. The acceleration potential gives directly the pressures. If one expresses the acceleration potential by a distribution of dipoles over the wing, then the local intensity of the dipoles gives the pressure difference between the upper and lower sides. In the wake there is no pressure difference, the unknown dipole distribution is applied only at the wing. In contrast there is a jump in the velocity potential not only at the wing but also between the upper and lower sides of the wake and one must allow for doublets over the surface of the wake too. This is no insurmountable obstacle, but in conjunction with the fact that the acceleration potential gives the desired pressure distribution directly it makes the acceleration potential attractive in subsonic flows. In a supersonic flow with a supersonic trailing edge the wake has no influence on the pressure distribution over the wing; but even with a subsonic trailing edge only a small portion of the wake has an influence. The advantage which one is

inclined to ascribe to the acceleration potential in a subsonic flow is greatly reduced in a supersonic flow.

3. The integral equations for both the acceleration and the velocity potential are obtained by representing the flow field by means of a dipole distribution, over the wing for the acceleration potential and over the wing and wake surface for the velocity potential. In discretizing the problem, one frequently divides the surface into elements and represents the dipole density in the individual elements by simple expressions, for instance by polynomials. At the element boundaries there will be some discontinuity in the first or higher derivatives. The upwash generated by such distributions will reflect these discontinuities. For both the acceleration and the velocity potential, discontinuities will occur at the element boundaries, but for the acceleration potential one has in addition discontinuities along the boundaries of the wake pertaining to that element. An upwash field arising in this manner is, therefore, traversed by lines of discontinuity in the direction of the x axis. This causes an uncertainty if one uses control points to match the upwash generated by the dipole distribution and the upwash given by the boundary conditions.

SECTION III

BASIC EQUATIONS FOR LINEARIZED STEADY SUPERSONIC FLOWS: LORENTZ TRANSFORMATION

Let $\bar{x},\bar{y},\bar{z}$ be a Cartesian System of coordinates, in which $\bar{x}$ has the free stream direction and the planform of the wing lies in the $\bar{x},\bar{y}$ plan. Let U be the stream velocity, M the supersonic free stream Mach number, $\beta = (M^2-1)^{1/2}$, L a characteristic length and $\bar{\phi}(\bar{x},\bar{y},\bar{z})$ the perturbation potential, which describes the deviation of the velocity field from a parallel flow. This potential is governed by the equation

$$-\beta^2\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{y}\bar{y}} + \bar{\phi}_{\bar{z}\bar{z}} = 0$$

We introduce dimensionless variables

$$\begin{aligned} \bar{x} &= xL & x &= \bar{x}/L \qquad (1)\\ \bar{y} &= yL & y &= \bar{y}/L \\ \bar{z} &= zL & z &= \bar{z}/L \end{aligned}$$

$$\bar{\phi}(\bar{x},\bar{y},\bar{z}) = UL\phi(x,y,z) = UL\phi(\bar{x}/L,\bar{y}/L,\bar{z}/L) \qquad (2)$$

One obtains

$$-\beta^2\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \qquad (3)$$

Originally the author had introduced at this point a Prandtl Glauert coordinate distortion by which the last equation is normalized to the case where $M = \sqrt{2}$. In view of a later extension to unsteady flows, where this distortion loses its usefulness, this is not done here. (It will be used in Appendix A, which deals with certain mathematical questions, but has no bearing on numerical work.)

At the planform the $\bar{z}$ component of the velocity is determined. Let the shape of the wing, including the effect of an angle of attack, be given by

$$\bar{z} = \bar{F}(\bar{x},\bar{y})$$

Then one has the boundary conditions

$$\bar{\phi}_{\bar{z}}(\bar{x},\bar{y},\bar{z}=0) = U\bar{F}_{\bar{x}}(\bar{x},\bar{y})$$

Hence in dimensionless form

$$\phi_z(x,y,z=0) = w(x,y) \tag{4}$$

with

$$w(x,y) = \bar{F}_{\bar{x}}(Lx,Ly) \tag{5}$$

A fundamental solution corresponding to a source (whose strength need not be defined) at a point $x = \xi$, $y = \eta$, $z = 0$ is given by

$$\phi^s(x,y,z) = [(x-\xi)^2 - \beta^2(\eta-y)^2 - \beta^2 z^2]^{-1/2}$$

provided that the point (x,y,z) lies in the aftercone of the point $(\xi, \eta, 0)$; outside the aftercone $\phi^s = 0$. The interior of the aftercone is given by

$$(x-\xi)^2 - \beta^2(\eta-y)^2 > \beta^2 z^2 \quad , \quad x>\xi$$

The flow over a wing with supersonic leading and trailing edges can be described by a distribution of such particular solutions over the planform

$$\phi^{(s)}(x,y,z) = \iint_A \frac{h(\xi,\eta)d\xi d\eta}{[(x-\xi)^2-\beta^2(\eta-y)^2-\beta^2 z^2]^{1/2}} \tag{6}$$

For given (x,y,z) the area of integration is the portion of the planform which lies within the forecone of the point x,y,z. Except for the leading edge the boundary of the region is then given by the curve of intersection of the forecone with the ξ,η-plane, i.e., by

$$(x-\xi)^2 - \beta^2(\eta-y)^2 - \beta^2 z^2 = 0$$

This is a hyperbola in the $\xi\eta$-plane. The asymptotes are the straight lines through the point $\xi = x$, $\eta = y$ with the slopes $\frac{d\eta}{d\xi} = \pm \beta^{-1}$. The vertex of the hyperbola lies at

$$\xi = x - \beta z$$

$$\eta = y$$

For $z=0$ the hyperbola degenerates into its asymptotes.

One can include problems with a subsonic leading edge in a procedure based on Eq. (6) (where the potential is expressed by a distribution of sources) by introducing a source distribution on a diaphragm extending from the leading edge to the Mach wave that forms the boundary of the region of influence of the wing. In the present report this possibility will not be explored. We shall use instead a representation of the flow field by a superposition of doublets.

In conjunction with the treatment of conical fields (Section VII) it will be useful to carry out a Lorentz transformation*

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*The idea to use a Lorentz transformation in the theory of linearized supersonic flow is fairly old. The author regrets that he is unable to give the reference in which it was first proposed.

$$\tilde{x} = (1-c^2)^{-1/2}x + \beta c(1-c^2)^{-1/2}y \qquad (7)$$

$$\tilde{y} = \beta^{-1}c(1-c^2)^{-1/2}x + (1-c^2)^{-1/2}y$$

$$x = (1-c^2)^{-1/2}\tilde{x} - \beta c(1-c^2)^{-1/2}\tilde{y} \qquad (8)$$

$$y = -\beta^{-1}c(1-c^2)^{-1/2}\tilde{x} + (1-c^2)^{-1/2}\tilde{y}$$

$$z = \tilde{z}$$

$$\phi(x,y,z)=\tilde{\phi}(\tilde{x},\tilde{y},\tilde{z})$$

The coordinates ξ and η are transformed in the same manner. Then

$$-\beta^2\phi_{xx} + \phi_{yy} + \phi_{zz} = -\beta^2\tilde{\phi}_{\tilde{x}\tilde{x}} + \tilde{\phi}_{\tilde{y}\tilde{y}} + \tilde{\phi}_{\tilde{z}\tilde{z}}$$

The hyperbola

$$(x-\xi)^2 - \beta^2(\eta-y)^2 - \beta^2z^2 = 0$$

transforms into

$$(\tilde{x}-\tilde{\xi})^2 - \beta^2(\tilde{\eta}-\tilde{y})^2 - \beta^2\tilde{z}^2 = 0 \qquad (9)$$

and one obtains for the Jacobian of the transformation

$$\frac{\partial(\tilde{x},\tilde{y})}{\partial(x,y)} = 1$$

After the Lorentz transformation one has

$$\tilde{\phi}(\tilde{x},\tilde{y},\tilde{z}) = \iint \frac{\tilde{h}(\tilde{\xi},\tilde{\eta})d\tilde{\xi}d\tilde{\eta}}{[\tilde{x}-\tilde{\xi})^2 - \beta^2(\tilde{\eta}-\tilde{y})^2 - \beta^2 z^2]^{1/2}} \tag{10}$$

with $\tilde{h}(\tilde{\xi},\tilde{\eta}) = h(\xi(\tilde{\xi},\tilde{\eta}),\eta(\tilde{\xi},\tilde{\eta}))$

The functions $\xi(\tilde{\xi},\tilde{\eta})$ and $\eta(\tilde{\xi},\tilde{\eta})$ are given by expression analogous to Equations (8). The boundary of the region of integration formed by the hyperbola, appears in the same form as before, i.e., one obtains Eq. (9). The boundary formed by the leading edge must be expressed in terms of the coordinates $\tilde{\xi}$ and $\tilde{\eta}$.

If, for instance, the leading edge is given by

$$y - c_1 x = 0, \text{ with } c_1 > \beta^{-1}$$

then one obtains in terms of $\tilde{x}$ and $\tilde{y}$

$$\tilde{y}(1+\beta c c_1) = \tilde{x}(c_1 + \beta^{-1}c)$$

Setting

$$c = -\frac{1}{\beta c_1} < 1$$

one obtains as equation of the leading edge

$$\tilde{x} = 0$$

This is a significant simplification.

SECTION IV

GENERAL REMARKS

For a wing of zero thickness and for boundary conditions satisfied at the planform, the prescribed z component of the velocity is symmetric with respect to the plane $z = 0$ (this is the plane of the planform). The potential is then antisymmetric. Outside the wing and the wake the potential is, therefore, zero; within the wing and the wake a jump of the potential from some value to the value with the opposite sign must be admitted. The pressure distribution over the wing is influenced only by those points of the wake for which part of the aftercone lies within the wing surface; because the effect of the upwash at any point of the flow field is felt only in its aftercone. For a supersonic trailing edge the wake has no effect on the wing. A potential which has the desired antisymmetry is obtained by a distribution of doublets (oriented in the z-direction) over the wing and, if necessary over part of the wake. A doublet potential is obtained by a differentiation of the source potential with respect to z. In formulating the boundary conditions at the wing one must express the z-component of the velocity field. One thus encounters a first differentiation with respect to z when one derives a doublet potential from the source potential and a second differentiation when one formulates the boundary conditions. The singularities which arise by these differentiations are the main concern of the following discussions. In the subsonic case these singularities occur only for $\xi = x$ and $\eta = y$. In contrast they occur in the supersonic case also along the hyperbola which form the boundary of the region of integration.

In the subsonic case one has the corresponding potential for a source

$$\phi = r^{-1}$$

with

$$r = [(x-\xi)^2 + \beta^2(\eta-y)^2 + \beta^2 z^2]^{1/2} \quad \text{with } \beta^2 = 1-M^2$$

Here r represents, in a suitable metric, the distance between the point (x,y,z) and the point (ξ,η,0). This can be carried over to the supersonic case if one introduces as distance definition

$$[(x-\xi)^2 - \beta^2(y-\eta)^2 - z^2]^{1/2}$$

The fundamental solutions are singular at all points where this "hyperbolic" distance is zero and this includes the points of the bounding hyperbola.

Incidentally, in a field with hyperbolic distance definition the Lorentz transformation is the counterpart to a rotation in a field with the elliptic distance definition.

SECTION V

THE TWO-DIMENSIONAL CASE

To familiarize ourselves with the mathematical technique we first treat the two-dimensional case. If one introduces the Lorentz transformation this includes the treatment of the infinitely long swept wing with a supersonic leading edge. For the two-dimensional case, the leading edge is always supersonic. The function h does not depend upon y and the evaluation of the potential can be carried out for $y = 0$. The upwash is needed for $z = 0$. One has

$$\phi(x,z) = \iint \frac{h(\xi)d\xi d\eta}{[(x-\xi)^2-\beta^2\eta^2-\beta^2 z^2]^{1/2}} \tag{11}$$

As was stated above the area of integration is bounded by the leading edge (here taken as the line $\xi = 0$) and the intersection of the forecone of the point (x,y,z) with the ξ,η plane. Because we carry out the integration for $y = 0$, the bounding hyperbola is given by

$$(x-\xi)^2 - \beta^2\eta^2 - \beta^2 z^2 = 0$$

In the limit $z = 0$ the boundary is given by its asymptotes

$$x - \xi \mp \beta\eta = 0$$

The vertex of the hyperbola lies at

$$x - \xi = \beta z, \qquad \eta = 0$$

With the limits specified, Eq. (11) appears in the form

$$\phi(x,0,z) = \int_0^{x-\beta z} h(\xi)\left(\int_{\eta_{lower}}^{\eta_{upper}} \frac{d\eta}{[(x-\xi)^2 - \beta^2 z^2 - \beta^2\eta^2]^{1/2}}\right)d\xi$$

where $\qquad \eta_{lower} = -[(x-\xi)^2 - \beta^2 z^2]^{1/2}/\beta$

$$\eta_{upper} = + [(x-\xi)^2 - \beta^2 z^2]^{1/2}/\beta$$

Setting

$$\eta = \bar{\eta}\beta^{-1}[(x-\xi)^2-\beta^2 z^2]^{1/2}$$

one obtains

$$[(x^2-\xi)^2 - \beta^2 z^2 - \beta^2\eta^2] = [(x^2-\xi)^2 - \beta^2 z^2]^{1/2}[1-\bar{\eta}^2]^{1/2}$$

and for the inner integral

$$\beta^{-1}\int_{-1}^{+1}\frac{d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}} = \beta^{-1}\pi$$

Hence $\phi(x,z) = \beta^{-1}\pi\int_{0}^{x-\beta z} h(\xi)d\xi$

$$\phi_x = \beta^{-1}\pi\ h(x-\beta z)$$

$$\phi_z = -\ \pi h(x-\beta z)$$

This satisfies the differential equation for one-dimensional flow

$$-\beta^2\phi_{xx} + \phi_{zz} = 0$$

SECTION VI

THE UPWASH IN THE GENERAL CASE

The point of departure is again Eq. (6). Let the leading edge be given by

$$\eta = g(\xi) \tag{12}$$

We set, in analogy to the procedure of the two-dimensional problems

$$\eta = y + \bar{\eta}\beta^{-1}q(x,\xi,z) \tag{13}$$

$$\bar{\eta} = \beta(\eta-y)/q$$

where

$$q(x-\xi,s) = [(x-\xi)^2-\beta^2 s]^{1/2} \tag{14}$$

To emphasize that in the expression q the coordinate z occurs only in the form of z^2, we have introduced $z^2=s$

One has $$q(x-\xi,0) = (x-\xi)$$

$$\frac{\partial q}{\partial x} = (x-\xi)/q \tag{15}$$

$$\frac{\partial q}{\partial z} = -\beta^2 z/q$$

The portion of the boundary of the region of integration formed by the hyperbola is transformed into

$$\bar{\eta} = \pm 1$$

The portion of the boundary formed by the leading edge appears as

$$\bar{\eta} = \bar{g}(x,y,\xi,s) \tag{16}$$

where

$$\bar{g}(x,y,\xi,s) = \frac{\beta(g(\xi)-y)}{q(x-\xi,s)} \tag{17}$$

The expression for ϕ generated by a source distribution is then given by

$$\phi^{(s)}(x,y,z) = \beta^{-1}\int_{\xi o}^{x-\beta z} \left(\int_{\bar{\eta}_{lower}}^{\bar{\eta}_{upper}} \frac{h(\xi,y+\bar{\eta}\beta^{-1}q)}{(1-\bar{\eta}^2)^{1/2}}\, d\bar{\eta}\right)d\xi \tag{18}$$

Here ξ_o is the smallest value of ξ in the region of integration. If the wing has a tip, ξ_o is independent of z. In cases where the foremost point of the region of integration is the intersection of the hyperbola with the leading edge, ξ_o will depend upon z. Furthermore,

$$\bar{\eta}_{\substack{lower\\upper}} = \mp 1 \text{ along the hyperbola} \tag{19}$$

$\bar{\eta}_{\substack{lower\\upper}} = \bar{g}(x,y,\xi,z)$ if the limit of $\bar{\eta}$ lies at the leading edge.

In Eq. (18) the limit $z \to 0$ can be formed immediately

$$\phi^{(s)}(x,y,z=0) = \beta^{-1}\int_{\xi o}^{x} \left(\int_{\bar{\eta}_{lower}}^{\bar{\eta}_{upper}} \frac{h(\xi,y+\bar{\eta}\beta^{-1}(x-\xi))}{(1-\bar{\eta}^2)^{1/2}} d\bar{\eta}\right)d\xi \tag{20}$$

This formulation is practical for supersonic leading and trailing edges. At the planform the upwash is given by Eq. (4). Accordingly, one must form in Eq. (18) the derivative $\phi_z^{(s)}(x,y,z=0)$. Before we do this let us determine $\phi_x^{(s)}$, which is needed to express the pressures in cases where one represents the flow field by a superposition of sources. If one is interested in only the pressures of the planform, then one can set z=0 before one carries out the differentiation, i.e., one can differentiate Eq. (20). The derivative with respect to the upper limit of the outer integral gives

$$\beta^{-1}h(x,y)\int_{-1}^{+1}\frac{d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}} = \beta^{-1}\pi h(x,y)$$

We denote by $h^{(1)}$ and $h^{(2)}$ the partial derivatives of h with respect to its first and second arguments. One then obtain

$$\phi_x^s(x,y,z=0) = \beta^{-1}h(x,y) + \beta^{-2}\int_{\xi o}^{x}\left(\int_{\bar{\eta}_{lower}}^{\bar{\eta}_{upper}}\frac{\bar{\eta}h^{(2)}(\xi,(y+\bar{\eta}\beta^{-1}(x-\xi))}{(1-\bar{\eta}^2)^{1/2}}d\bar{\eta}\right)d\xi \qquad (21)$$

The limits for $\bar{\eta}$ are determined for z=0. Hence from Eqs. (19) and (13) and (14)

$$\bar{\eta}_{\substack{lower\\upper}} = \mp 1$$

or

$$\bar{\eta}_{\substack{lower\\upper}} = \beta\frac{g(\xi)-y}{x-\xi}$$

Next the derivatives of $\phi^{(s)}$ with respect to z are evaluated. As we mentioned above, the first derivative is needed in satisfying the upwash conditions, if one expresses the potential by a source distribution. Besides, the expression $\phi_z^{(s)}$ can be interpreted as the potential due to a doublet distribution.

$$\phi^d(x,y,z) = \phi_z^s(x,y,z)$$

To formulate the upwash conditions for the potential ϕ^d one must form

$$\phi_z^d(x,y,z) = \phi_{zz}^s(x,y,z)$$

Ultimately, all quantities will be evaluated for z=0. Configurations of the region of integration are shown in Figures 1 and 2. The leading edge may contain subsonic or supersonic parts. For supersonic or subsonic parts of the leading edge the slope $d\eta/d\xi = dg/d\xi$ is respectively greater or smaller than β^{-1}. A leading edge which is entirely subsonic consists of two parts, one with positive the other with negative slope. We divide the region of integration into two parts I and II by a straight line $\xi=\xi_1$ lying in the region where the upper and lower limits of $\bar{\eta}$ are ± 1. The contributions to the potential from these regions are denoted by $\phi^{(s,I)}$ and $\phi^{(s,II)}$ or $\phi^{(d,I)}$ and $\phi^{(d,II)}$, (if they are generated respectively by sources or doublets), Figure 3.

Consider

$$\phi^{(s,I)}(x,y,z) = \beta^{-1}\int_{\xi_1}^{x-\beta z} \left(\int_{-1}^{+1} \frac{h(\xi, y+\bar{\eta}\beta^{-1}q)\,d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}}\right) d\xi$$

Then by differentiation with respect to z

$$\phi_z^{(s,I)}(x,y,z) = -\int_{-1}^{+1} \frac{h(\xi,y+\bar{\eta}\beta^{-1}q)d\bar{\eta}}{(1-\eta^2)^{1/2}} \Bigg|_{\xi=x-\beta z}$$

$$+\ \beta^{-2}\int_{\xi_1}^{x-\beta z} \left(\int_{-1}^{+1} \frac{h^{(2)}(\xi,y+\bar{\eta}\beta^{-1}q)\ \bar{\eta}(\partial q/\partial z)d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}} \right) d\xi \qquad (22)$$

Because $$q(x-\xi,z)\Big|_{\xi = x-\beta z} = 0$$

the first term gives immediately

$$-h(x-\beta z,y)\int_{-1}^{+1} \frac{d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}} = -\ \pi h(x-\beta z,y)$$

Substituting $\frac{\partial q}{\partial z}$ from Eq. (15) one obtains for the second term

$$-\ z\int_{\xi_1}^{x-\beta z} \left(\int_{-1}^{+1} \frac{h^{(2)}(\xi,(y+\bar{\eta}\beta^{-1}q))\bar{\eta}d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}} \right) \frac{1}{q(x-\xi,s)} d\xi$$

The factor q tends to zero as ξ approaches its upper limit and for z=0 (s=0) one obtains a non-convergent integral of the form $\int \frac{d\xi}{x-\xi}$. One observes, however, that

$$\int_{-1}^{+1} \frac{h^{(2)}(\xi,y)\bar{\eta}d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}} = 0$$

because of the antisymmetry of the integrand with respect to $\bar{\eta}$. The inner integral can, therefore, be replaced by

$$\int_{-1}^{+1} \frac{[h^{(2)}(\xi,(y+\bar{\eta}\beta^{-1}q))-h^{(2)}(\xi,y)]\bar{\eta}d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}}$$

Let

$$\frac{h^{(2)}(\xi,y+\zeta)-h^{(2)}(\xi,y)}{\zeta} = \psi_1(\xi,y,\zeta)$$

The function exists even for $\xi=0$. There

$$\psi_1(\xi,y,0) = h^{(2,2)}(\xi,y)$$

(where $h^{(2,2)}$ denotes the second partial derivative with respect to the second argument of h.)

Then

$$h^{(2)}(\xi,y+\bar{\eta}\beta^{-1}q) - h^{(2)}(\xi,y) = \bar{\eta}\beta^{-1}q\psi_1(\xi,y,\bar{\eta}\beta^{-1}q)$$

and the expression $\phi_z^{(s,I)}$ assumes the form

$$\phi_z^{(s,I)} = -\pi h(x-\beta z,y)-z\beta^{-1}\int_{\xi_1}^{x-\beta z}\left(\int_{-1}^{+1}\frac{\psi_1(\xi,y+\bar{\eta}\beta^{-1}q)\bar{\eta}^2d\bar{\eta})}{(1-\bar{\eta}^2)^{1/2}}\right)d\xi$$

The inner integral is bounded. In the limit z=0 the second term vanishes and one obtains

$$\phi^{(d,I)}(x,y,z=0) = \phi_z^{(s,I)}(x,y,z=0) = -\pi h(x,y) \qquad (23)$$

The contribution of the region I to the second derivative of $\phi^{(s)}$ with respect to z is

$$\phi_z^{(d,I)}(x,y,z) = \pi\beta h^{(1)}(x-\beta z,y)$$

$$-\beta^{-1}\int_{\xi_1}^{x-\beta z}\left(\int_{-1}^{+1}\frac{\psi_1(\xi,y,\bar{\eta}\beta^{-1}q))\bar{\eta}^2 d\bar{\eta})}{(1-\bar{\eta}^2)^{1/2}}\right)d\xi$$

$$+z\int_{-1}^{+1}\frac{\psi_1(\xi,y,\bar{\eta}\beta^{-1}q))\bar{\eta}^2 d\bar{\eta})}{(1-\bar{\eta}^2)^{1/2}}\Bigg|_{\xi=x-\beta z}$$

$$-z\beta^{-2}\int_{\xi_1}^{x-\beta z}\left(\int_{-1}^{+1}\frac{\psi_1^{(3)}(\xi,\bar{y},\bar{\eta}\beta^{-1}q)\bar{\eta}^3(\partial q/\partial z)}{(1-\bar{\eta}^2)^{1/2}}\,d\bar{\eta}\right)d\xi$$

Here $\psi_1^{(3)}$ denotes the partial derivative of ψ_1 with respect to its third argument. The second term on the right has already been evaluated, it gives a non-vanishing contribution. The third term vanishes in the limit $z=0$, because of the factor z. The fourth term has actually a factor z^2, because $\frac{\partial q}{\partial z} = -\beta^2 z/q$. Thus, it will vanish in the limit $z\to 0$ provided that the remaining expression remains finite. One has after substitution of $\partial q/\partial z$

$$-z^2\int_{\xi_1}^{x-\beta z}\left(\int_{-1}^{+1}\frac{\psi_1^{(3)}(\xi,y,\bar{\eta}\beta^{-1}q))\bar{\eta}^3 d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}}\right)\frac{d\xi}{q}$$

Again one is concerned about the factor q in the denominator which vanishes at the upper limit $\xi=x-\beta z$. Here the procedure shown above is applied again. One observes that

$$\int_{-1}^{+1}\frac{\psi_1^{(3)}(\xi,y,0)\bar{\eta}^3 d\eta}{(1-\bar{\eta}^2)^{1/2}} = 0$$

because of the antisymmetry of the integrand with respect to $\bar{\eta}=0$.

One introduces

$$\psi_2(\xi,y,\xi) = \frac{\psi_1^{(3)}(\xi,y,\zeta)-\psi_1^{(3)}(\xi,\eta,0)}{\zeta}$$

If $\psi_1^{(3)}$ is sufficiently smooth, as it is except for possible singular lines, the function exists for $\zeta=0$.

Therefore,

$$\frac{1}{q}\int_{-1}^{+1}\frac{\psi_1^{(3)}(\xi,y,(\bar{\eta}\beta^{-1}q)\bar{\eta}^3 d\eta}{(1-\bar{\eta}^2)^{1/2}} = \beta^{-1}\int_{-1}^{+1}\frac{\psi_2(\xi,y,(\bar{\eta}\bar{\beta}^1 q)\bar{\eta}d\eta}{(1-\bar{\eta}^2)^{1/2}}$$

and this expression exists.

One then obtains

$$\phi_z^{(d,I)}(x,y,z=0) = \pi\beta h^{(1)}(x,y)-\beta^{-1}\int_{\xi_1}^{x}\left(\int_{-1}^{+1}\frac{\bar{\eta}^2\psi_1(\xi,y,\bar{\eta}\beta^{-1}(x-\xi))d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}}\right)d\xi$$

Substituting ψ_1, one obtains for the second term

$$-\int_{\xi_1}^{x}\left(\int_{-1}^{+1}\frac{\bar{\eta}h^{(2)}(\xi,y+\bar{\eta}\beta^{-1}(x-\xi))d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}(x-\xi)}\right)d\xi \qquad (24)$$

As ξ approaches the upper limit x the integrand seems to tend to infinity but by the above argument one obtains a finite expression for the inner integral provided that the integrations are carried out in the sequence shown in Eq. (24). To allow for a change in the sequence of integrations we write

$$I = -\lim_{\varepsilon\to 0}\int_{\xi_1}^{x-\varepsilon}\left(\int_{-1}^{+1}\frac{\bar{\eta}h^{(2)}(\xi,\ y+\bar{\eta}\beta^{-1}(x-\xi))d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}(x-\xi)}\right)d\xi$$

Now it is permissible to change the sequence of integrations

$$I = -\lim_{\varepsilon\to 0}\int_{-1}^{+1}\frac{\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}}\left(\int_{\xi_1}^{x-\varepsilon}\frac{h^{(2)}(\xi,(y+\bar{\eta}\beta^{-1}(x-\xi))d\xi}{x-\xi}\right)d\bar{\eta} \qquad (25)$$

According to this formulation one first excludes the neighborhood of the point $(\xi,\eta) = (x,y)$ by a straight line $\xi = x-\varepsilon$ $(\varepsilon>0)$, and after the integrations forms the limit $\varepsilon\to 0$. Then

$$\phi_z^{(d,I)}(x,y,z=0) = \pi\beta h^{(1)}(x,y) + I \qquad (26)$$

where I is evaluated either in the form of Eq. (24) or of Eq. (25). The regions of integrations for $z=0$ and $z\neq 0$ are shown in Figure 4.

In the region II, $x-\xi>0$ therefore, $q=[(x-\xi)^2-\beta^2 z^2]^{1/2}>0$, even for $z=0$. A denominator q is no longer detrimental. But here part of the boundary of the region depends, after the transformation to $\bar{\eta}$, upon $s=z^2$. (The transformation to $\bar{\eta}$ is desirable because of those portions of the boundary formed by the hyperbola.) We interchange the sequence of integrations in Eq. (18):

$$\phi^{(s,II)}(x,y,z) = \beta^{-1}\int_{-1}^{+1}\frac{1}{1-\bar{\eta}^2)^{1/2}}\left(\int_{\bar{f}(x,y,\bar{\eta},s)}^{\xi_1} h(\xi,y+\bar{\eta}\beta^{-1}q)d\xi\right)d\bar{\eta} \qquad (27)$$

Here $\bar{f}(x,y,\bar{\eta},s)$ denotes the inverse of the function $\bar{g}(x,y,\xi,s)$, Eq. (17), at fixed x,y and s. Then

$$\phi_z^{(s,II)}(x,y,z) = \beta^{-1}\int_{-1}^{+1} \frac{1}{(1-\bar{\eta}^2)^{1/2}}\{-2z\frac{\partial\bar{f}}{\partial s}\, h(\xi,y+\bar{\eta}\beta^{-1}q)\Big|_{\xi=\bar{f}(x,y,\bar{\eta},s)}\}d\bar{\eta} \tag{28}$$

$$-z\int_{-1}^{+1} \frac{\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}}\left(\int_{\bar{f}(x,y,\bar{\eta},s)}^{\xi_1} \frac{h^{(2)}(\xi,y+\bar{\eta}\beta^{-1}q)d\xi}{q}\right)d\bar{\eta}$$

In the first expression h is evaluated, for the particular value of $\bar{\eta}$ under consideration, at the lower limit of the region of integration over ξ. This is a point of the leading edge and there $h=0$. For $z=0$ both terms on the right of the last equation vanish

$$\phi_z^{(s,II)}(x,y,z=0) = 0$$

We combine this result with Eq. (23) and obtain

$$\phi_z^s(x,y,z=0^+) = \phi^d(x,y,z=0) = -\pi h(x,y) \tag{29}$$

We now turn to the second derivatives of $\phi^{(s)}$ with respect to z. At the leading edge the potential and with it h is zero and continuous. The first term on the right of Eq. (28) is, therefore, zero, even for $z\neq 0$. Differentiating the second term with respect to z, one obtains an expression which will not vanish for $z=0$, namely

$$-\int_{-1}^{+1} \frac{\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}}\left(\int_{\bar{f}(x,y,\bar{\eta},0)}^{\xi} \frac{h^{(2)}(\xi,y+\bar{\eta}\beta^{-1}q(x-\xi,0)}{q(x-\xi,0)}d\xi\right)d\bar{\eta} \tag{30}$$

Further terms arise by differentiations of s with respect to z; they introduce other factors z. One will surmise, that these terms vanish for $z=0$. This, however, is not self-evident. At a subsonic leading edge h behaves as $u^{1/2}$, where u is the distance from the leading edge. The differentiation with respect to z

generates some terms which are infinite as ξ approaches the leading edge. (One of these is $h^{(2)}$ itself.) Actually, they will cancel. A detailed discussion is carried out in Appendix A. Eq (30) is indeed the only contribution to $\phi_z^{(d,II)}(x,y,z=0)$. Anticipating these results, we compile here the formulae needed for a numerical approach.

$$\phi^d(x,y,0^+) = -\pi h(x,y)$$

$$\phi_z^d(x,y,0) = \pi\beta h^{(1)}(x,y) - \iint_{A'} \frac{\bar{\eta}\, h^{(2)}(\xi(y+\bar{\eta}\beta^{-1}(x-\xi)))}{(1-\bar{\eta}^2)^{1/2}(x-\xi)} d\xi d\bar{\eta} \qquad (31)$$

where A' is the area of the planform within the fore-cone of the point $(x,y,z=0)$, but with the point $(\xi,\eta)=(x,y)$ excluded by a line $\xi=x-\varepsilon$, $\varepsilon>0$. Subsequently, one forms the limit $\varepsilon\to 0$. The same expression is written in the original coordinates. Equating the resulting expression with the given downwash $w(x,y)$, one obtains the integral equation from which the doublet density $h(x,y)$ can be determined. The limiting process $z\to 0$ no longer appears in this equation.

$$w(x,y) = \phi_z^d(x,y,0) = \pi\beta h^{(1)}(x,y) - \beta^2 \iint_{A'} \frac{(\eta-y)h^{(2)}(\xi,\eta)d\xi d\eta}{[x-\xi)^2-\beta^2(\eta-y)^2]^{1/2}(x-\xi)^2} \qquad (32)$$

If the integration with respect to η is carried out first, then the exclusion of the point $(\xi,\eta) = (x,y)$ is not necessary. But if the integrations are carried out in a different sequence then in some intermediate steps one obtains contributions which are large if ξ approaches x and the precautionary step of introducing A', which initially excludes the critical vicinity of the point (x,y), is necessary.

Eq. (31) expresses the upwasn in terms of $h^{(1)} = h_x$ at the point (x,y) and of an integral which is due to the deviation from a two-dimensional flow. (One notices that the integrand contains a factor $h^{(2)} = h_\eta$.) Because of the denominator

$[(x-\xi)^2 - \beta^2(\eta-y)^2]^{1/2}$, points in the vicinity of the Mach waves through the point (x,y) enter with greater weight. A good approximation for the vicinity of these lines is, therefore, desirable.

Some thoughts about the application of Eq. (32) in a numerical procedure are found in Section VIII.

All transformation carried out in this Section could have been performed in Eq. (10). Therefore, one obtains alternatively

$$\tilde{w}(\tilde{x},\tilde{y}) = \tilde{\phi}_z^d(\tilde{x},\tilde{y},0) = \pi\beta\tilde{h}_{\tilde{x}}(\tilde{x},\tilde{y}) - \beta^2 \iint \frac{(\tilde{\eta}-\tilde{y})(\partial(\tilde{h}(\tilde{\xi},\tilde{\eta})/d\tilde{\eta})d\tilde{\xi}d\tilde{\eta}}{[(\tilde{x}-\tilde{\xi})^2-\beta^2(\tilde{\eta}-\tilde{y})^2]^{1/2}(\tilde{x}-\tilde{\xi})^2} ,$$

$\tilde{h}(\tilde{x},\tilde{y})$ has been defined in conjunction with Eq. (10).

Moreover

$$\tilde{w}(\tilde{x},\tilde{y}) = w(x(\tilde{x},\tilde{y}),y(\tilde{x},\tilde{y})).$$

SECTION VII

CONICAL FIELDS

Frequently, the planform has a tip at which two straight subsonic leading edges meet. In the vicinity of this tip one obtains a conical field. The original concept of conical fields is applicable if the angle of attack is constant. The wing is then regarded as a degenerate cone generated by straight lines through the tip lying in a plane. In this case the velocity vector is constant along each straight line through the tip. Basically, one deals with a similarity solution, which reduces the number of dimensions of the problem from three to two.

In linearized flow the concept of a conical field is applicable also if the upwash behaves like a power of the distance from the tip. Separately for each such power, one makes a similarity hypothesis. If the velocities are constant along straight lines through the tip one can apply a further transformation to the resulting partial differential equations in two dimensions (for the individual velocity components) which yields the Laplace equation. In this case the theory of analytic functions can be applied. Unfortunately, one is led, at least in the present problem, to the evaluation of elliptic integrals. Moreover, if the velocity is not constant but follows a general power law, then the theory of analytic functions is not applicable. We show here, how such conical fields can be treated (numerically) by means of the present integral equation formulation.

The procedure is first described in general terms. Let the leading edges be given by

$$y/x = a_1 \text{ and } y/x = -a_2 \quad ; \quad a_1 > 0,\ a_2 > 0$$

The planform is nearly always symmetric with respect to the x axis. Then $a_1 = a_2$. For subsonic edges $a_1 < \beta^{-1}$, $a_2 < \beta^{-1}$.

The present approach assumes that the upwash is given by a linear combination of expressions

$$w(x,y) = x^m \hat{w}^{(m)}(y/x) \qquad (33)$$

In practice one usually has

$$w(x,y) = x^{\tilde{m}} y^j$$

Then

$$m = \tilde{m}+j$$

$$\hat{w}(y/x) = (y/x)^j \qquad (34)$$

The computations are carried out separately for each choice of m and j.

For a constant angle of attack m = 0 and $\hat{w}^{(m)}$ is constant. In most practical applications m = 0, 1 or 2 will be sufficient.

The unknown function h is now written in the form

$$h(x,y) = \sum_n b_n h^{mn}(y/x) \qquad (35)$$

where

$$h^{mn} = x^{m+1} h^n(y/x)$$

The functions $h^n(y/x)$ are known. Probably, one will use the same functions h^n for all values of m. The coefficients b_n are to be determined. Substituting Eq. (35) into the integral equation, Eq. (32), one obtains

$$x^m \hat{w}^{(m)}(y/x) = \sum_n b_n Q^{mn}(y/x) \qquad (36)$$

where $Q^{mn}(y/x)$ is the upwash generated by a function $h^{mn} = x^m h^n(y/x)$. This substitution gives

$$Q^{mn}(y,x) = Q^{mn1} + Q^{mn2}$$

with

$$Q^{mn1} = \beta\pi(\partial h^{mn}(x,y)/\partial x)$$

$$Q^{mn2} = -\beta^2 \iint \frac{(\eta-y)(\partial h^{mn}(\xi,\eta)/\partial\eta)d\xi d\eta}{[(x-\xi)^2-\beta^2(\eta-y\ )^2]^{1/2}(x-\xi)^2} \tag{37}$$

In discretizing the problem one uses only a finite number of functions $h^n(y/x)$. To determine the coefficients b_n a collocation method is probably sufficient. We set for this purpose $y/x = c_k$ and admits as many values of c_k as one uses values of n. Then one obtains the following system of equations

$$x^m Q^{(m)}(c_k) = \sum_1^N b_n Q^{mn}(c_k) \quad , \quad k = 1,..N \tag{38}$$

The main task is the determination of the matrix elements $Q^{mn}(c_k)$.

Regarding the cnoice of the functions h^{mn} we make the following observation. If the angle at the tip is small, and $a_1=a_2=a$, then one obtains from slender body theory, that the potential, and with it, h is given by

$$\text{const } (a^2x^2 - y^2)^{1/2} \quad ; \; y < ax,$$

This suggests that in a more general situation h can be represented by

$$h = \Sigma\ b_n\ h^{mn}(y/x) = \Sigma\ b_n x^{m+1} h^n(y/x) \tag{39}$$

with

$$h^n(y/x) = [(a_1-(y/x)(a_2+(y/x))]^{1/2}(y/x)^n \tag{40}$$

Since for a slender tip the first term is already sufficient it can be expected that the number of terms necessary to obtain a

reasonable accuracy is fairly small. Moreover, if $a_1 = a_2$ and the upwash is either symmetric or antisymmetric with respect to the x axis, then only the values of n which respectively are even or odd will occur.

The leading edges are given by

$$y/x = a_1 \;;\; 0 < a_1 < \beta^{-1}$$

and

$$y/x = -a_2 \;;\; 0 < a_2 < \beta^{-1}$$

The further development is suggested by the following observation. The expression Q^{mn1} defined in Eq. (37), for $m = 0$ and for

$$h = (a^2x^2 - y^2)^{1/2}$$

contains an essential term

$$\partial h/\partial x = a^2(a^2 - (y/x)^2)^{-1/2}$$

This expression and, therefore, Q^{mn1} tends to infinity as y/x tends to a. But one expects from slender body theory that the upwash caused by this expression is finite over the entire width of the airfoil. The expression Q^{mn2} will, therefore, contain some term which gives an equally large contribution but with the opposite sign, so that this infinity is cancelled. This complication can be avoided if one first carries out a Lorentz transformation (separately for each of the chosen values y/x) so that the chosen value of y/x is transformed into $\tilde{y}/\tilde{x} = 0$.

The Lorentz transformation is first carried out without this specialization in order to provide a clear picture of the individual steps. We repeat the transformation formulae

$$x = (1-c^2)^{-1/2} \, (\quad \tilde{x} \quad - \beta c\tilde{y}) \tag{41a}$$

$$y = (1-c^2)^{-1/2} \, (-\beta^{-1} c\tilde{x} + y)$$

$$\tilde{x} = (1-c^2)^{-1/2} \, (\quad x \quad + \beta c y) \tag{41b}$$

$$\tilde{y} = (1-c^2)^{-1/2} \, (\beta^{-1} c x \quad + y)$$

The variables ξ and η are transformed in the same manner. By the Lorentz transformation the left hand side of Eq. (38) assumes the form

$$(1-c^2)^{-m/2} (\tilde{x}-\beta c\tilde{y})^m \hat{W}^m \left(\frac{-\beta^{-1} c+(\tilde{y}/\tilde{x})}{1-\beta c(y/x)}\right)$$

The constant c will be chosen in such a manner, that, $\tilde{y} = 0$ for $y/x = c_k$. From the second of Eq. (41b) one then obtains

$$c = -\beta c_k \tag{42}$$

With this choice one obtains for the left hand side of Eq. (38)

$$(1-c^2)^{-m/2} \, \tilde{x}^m \hat{W}^m (c_k) \tag{43}$$

To transform the right hand side of Eq. (38) one must first carry out the transformation to $\tilde{\xi}$ and $\tilde{\eta}$ in

$$h^{mn}(x,y) = x^{m+1} [(a_1-(y/x))(a_2+(y/x))]^{1/2} (y/x)^n \quad ; \tag{44}$$

$\tilde{h}^{mn}(\tilde{\xi},\tilde{\eta})$ is obtained by substituting $x = x(\tilde{x},\tilde{y})$ and $y = y(\tilde{x},\tilde{y})$.

Let $$\beta\tilde{\eta}/\tilde{\xi} = p \tag{45}$$

Then, from Eq. (41a)

$$\eta/\xi = \frac{p-c}{\beta(1-cp)} \tag{46}$$

Moreover

$$\xi = (1-c^2)^{-1/2}\tilde{\xi}(1-cp) \tag{47}$$

The leading edges are originally given by

$$\eta/\xi = a_1 \text{ and } \eta/\xi = -a_2$$

One finds from Eq. (41)

$$\tilde{\eta}/\tilde{\xi} = \frac{\beta^{-1}c+(y/x)}{1+\beta cy/x}, \tag{48}$$

The leading edges, therefore, transform into

$$p=\tilde{a}_1 \text{ and } p = -\tilde{a}_2 \tag{49}$$

with

$$\tilde{a}_1 = \frac{\beta a_1+c}{1+c\beta a_1}, \quad \tilde{a}_2 = \frac{\beta a_2-c}{1+c\beta a_1} \tag{50}$$

In the expression h^n Eq. (40), a factor

$$[(a_1-\eta/\xi)(a_2+\eta/\xi)]^{1/2}$$

occurs. One finds with Eq. (45)

$$(a_1-\eta/\xi) = \frac{1+\beta a_1 c}{\beta(1-cp)}(\tilde{a}_1-p) \tag{51}$$

$$(a_2-\eta/\xi) = \frac{1-\beta a_2 c}{\beta(1-cp)}(\tilde{a}_2+p) \tag{52}$$

Now the transformation of w^{mn}, Eq. (44), can be carried out

$$\tilde{h}^{mn}(\tilde{\xi},\tilde{\eta}) = (1-c^2)^{-m/2}\, a_3\, \beta^{-n-1}\tilde{\xi}^{m+1}\, F^{mn}(p) \tag{53}$$

with

$$a_3 = (1-c^2)^{-1/2}[(1+\beta a_1 c)(1-\beta a_2 c)]^{1/2} \tag{54}$$

$$F^{mn}(p) = [(\tilde{a}_1-p)(\tilde{a}_2+p)]^{1/2}(1-cp)^{m-n}(p-c)^n \tag{55}$$

where p is expressed in terms of $\tilde{\eta}/\tilde{\xi}$ by Eq. (45). We note

$$dF^{mn}/dp = [(\tilde{a}_1-p)(\tilde{a}_2+p)]^{-1/2}\ f^{mn}(p) \tag{56}$$

where

$$\begin{aligned} f^{mn}(p) = \left(-p+\frac{\tilde{a}_1-\tilde{a}_2}{2}\right)&(1-cp)^{m-n}(p-c)^n \\ +(\tilde{a}_1-p)(\tilde{a}_2-p)&[(-c)(m-n)(1-cp)^{m-n-1}(p-c)^n \\ &+ n(1-cp)^{m-n}(p-c)^{n-1}] \end{aligned} \tag{57}$$

The second term in the last bracket vanishes for n=0, therefore no difficulty arises, even for p=c.

One has in particular

$$f^{mn}(0)=(\tilde{a}_1-\tilde{a}_2)(-c)^n + \tilde{a}_1\ \tilde{a}_2[(m-n)(-c)^{n+1}+n(-c)^{n-1}] \tag{58}$$

Moreover

$$\begin{aligned} (df^{mn}/dp)\Big|_{p=0} = -(-c)^n &+ (3/2)(\tilde{a}_1-\tilde{a}_2)[(m-n)(-c)^{n+1}+n(-c)^{n-1}] \\ &+\tilde{a}_1\tilde{a}_2[(m-n-1)(m-n)(-c)^{n+2}+2(m-n)n(-c)^n \\ &+ n\ (n-1)(-c)^{n-2}] \end{aligned} \tag{59}$$

One thus obtains for general c, by substituting Eq. (53)

$$\tilde{Q}^{mn}(\tilde{x},\tilde{y}) = (1-c^2)^{-m/2}\ a_3\ \{\beta^{-n}\pi\partial(\tilde{x}^{m+1}F^{mn}(\beta\tilde{y}/\tilde{x}))/\partial\tilde{x})$$

$$- \beta^{-n+1}\iint\frac{(\tilde{\eta}-\tilde{y})(\partial(\tilde{\xi}^{m+1}F^{mn}(\beta\tilde{\eta}/\tilde{\xi}))/\partial\tilde{\eta})d\tilde{\xi}d\tilde{\eta}}{[(\tilde{x}-\tilde{\xi})^2-\beta^2(\tilde{\eta}-\tilde{y})^2]^{1/2}(\tilde{x}-\tilde{\xi})^2}\}$$

To make the dependence upon $\tilde{\xi}$, $\tilde{\eta}$ and $\tilde{x}$, $\tilde{y}$ obvious we have here substituted the expression for p. With the special choice made above, namely $c = -\beta c_k$, one obtains $\tilde{y} = 0$ for $y/x = c_k$. Then the expression simplifies to

$$\tilde{Q}^{mn}(\tilde{x},0) = (1-c^2)^{-m/2}\tilde{x}^m\hat{Q}^{mn} \tag{60a}$$

with

$$\hat{Q}^{mn} = \hat{Q}^{mn1} + \hat{Q}^{mn2} \tag{60b}$$

where

$$\hat{Q}^{mn1} = a_3\beta^{-n}\tilde{x}^{-m}\partial(\tilde{x}^{m+1}F^{mn}(0))/\partial\tilde{x} \tag{61}$$

$$\hat{Q}^{mn2} = -a_3\beta^{-n+1}\tilde{x}^{-m}\iint\frac{\tilde{\eta}\partial(\tilde{\xi}^{m+1}F^{mn}(\beta\tilde{\eta}/\tilde{\xi}))/\partial\tilde{\eta})d\tilde{\xi}d\tilde{\eta}}{[(\tilde{x}-\tilde{\xi})^2 - \beta^2\tilde{\eta}^2]^{1/2}(\tilde{x}-\tilde{\xi})^2} \tag{62}$$

Substituting Eqs. (43) and (60) into Eq. (38) one obtains

$$\hat{w}^m(c_k) = \sum_1^N b_n\hat{Q}^{mn}(c_k) \tag{63}$$

In $\hat{Q}^{mn1}$, Eq. (61), $F^{mn}(0)$ from Eq. (55) is substituted and the differentiation with respect to $\tilde{\xi}$ is carried out.

Then

$$\hat{Q}^{mn1} = a_3\beta^{-n}(m+1)(\tilde{a}_1\tilde{a}_2)^{1/2}(-c)^n \tag{64}$$

In the expression $\hat{Q}^{mn2}$, Eq. (62), p defined in Eq. (45), is introduced as variable of integration instead of $\tilde{\eta}$. Then

$$d\tilde{\eta} = \beta^{-1}\tilde{\xi}dp \qquad (65)$$

In addition, we set

$$\tilde{\xi}/\tilde{x} = \hat{\xi}. \qquad (66)$$

The region of integration is a quadrangle. Two sides are formed by the leading edges. The lines p = const are straight lines through the tip of the airfoil. The limits for p are then according to Eqs. (49) $p = -\tilde{a}_2$ and $p = \tilde{a}_1$.

The other two sides are given by the asymptotes of the original hyperbolic boundary, into which this boundary deforms as $z \to 0$. They are the intersection of the characteristic cone through the point x_o, y_o with the planform. The Lorentz transformation leaves the equations for the hyperbola, and therefore, also for its asymptotes unchanged; one therefore has for the other boundaries

$$\beta\tilde{\eta} + \tilde{\xi} = \tilde{x}_o \quad \text{for } \tilde{\eta} > 0$$

$$\beta\tilde{\eta} - \tilde{\xi} = \tilde{x}_o \quad \text{for } \tilde{\eta} < 0.$$

One, therefore, obtains in either case

$$\tilde{\xi} = \frac{\tilde{x}_o}{1+|p|}$$

or

$$\hat{\xi} = \frac{1}{1+|p|}$$

Now the expression $\hat{Q}^{mn2}$, Eq. (62) can be rewritten. We express $dp/d\tilde{\eta}$ by Eq. (65) and dF^{mn}/dp by Eq. (56). Carrying out the integration with respect to p last, one obtains

$$\hat{Q}^{mn2} = \pi a_3 \beta^{-n} \int_{-\tilde{a}_2}^{\tilde{a}_1} p[(\tilde{a}_1 - p)(\tilde{a}_2 + p)]^{-1/2} f^{mn}(p) I^m(p) dp \tag{67}$$

with

$$I^m(p) = \int_0^{(1+|p|)^{-1}} \frac{\hat{\xi}^{m+2} d\hat{\xi}}{(1-\hat{\xi})^2 [(1-\hat{\xi})^2 - p^2 \hat{\xi}^2]^{1/2}} \tag{68}$$

One notices that the integral $I^m(p)$ does not depend upon n. The integrand has only two second order branch points; $I^m(p)$ can therefore be evaluated analytically. The integration needed in Eq. (67) is best carried out numerically.

It was mentioned above the values of m are determined by the character of the upwash, m=0, 1, 2 will probably be sufficient for all practical purposes. The evaluation of the integrals (68) is shown in Appendix B. Here we give the results of m=0, 1 and 2.

Let

$$g^{(0)}(p^2) = (1-p^2)^{-1/2}$$

$$g^{(1)}(p^2) = (1-p^2)^{-3/2}(3-2p^2) \tag{69}$$

$$g^{(2)}(p^2) = (1-p^2)^{-5/2}((6-(15/2)p^2+3p^4)$$

and

$$\bar{g}^{(0)} = 0$$

$$\bar{g}^{(1)} = -(1-p^2)^{-1}$$

$$\bar{g}^{(2)} = (-(7/2) + 2p^2)(1-p^2)^{-2}$$

Let $$I^m(p) = I_a^m(p) + I_b^m(p) + I_c^m(p) \tag{70}$$

$$I_a^m(p) = p^{-2} + g^{(m)}(p^2)(\log(1+(1-p^2)^{1/2}) + \bar{g}^{(m)}$$

$$I_b^m(p) = -\pi \, \frac{m+1}{2} \, p^{-1} \tag{71}$$

$$I_c^m(p) = -g^m(p^2)\log|p|$$

The distinction between I_a^m, I_b^m, I_c^m, has been made, because they must be treated separately in the integration with respect to p. This is obvious from the terms p^{-2}, $|p|^{-1}$ and $\log|p|$ in $I_a^{(0)}$, $I_b^{(0)}$, and $I_c^{(0)}$, respectively. Accordingly, we write for the expression (67)

$$\hat{Q}^{mn2} = Q_a^{mn2} + Q_b^{mn2} + Q_c^{mn2} \tag{72}$$

where the right hand sides arise by replacing $I^m(p)$ by the respective expression $I_a^m(p)$, $I_b^m(p)$ and $I_c^m(p)$. One has specifically

$$Q_a^{mn2} =$$

$$-a_3\beta^{-n}\int_{-\tilde{a}_2}^{\tilde{a}_1}[(\tilde{a}_1-p)(\tilde{a}_2+p)]^{-1/2}[p^{-1}+pg^{(m)}\log(1+(1-p^2)^{1/2})+p\bar{g}^{(m)}]f^{mn}(p)dp$$

The term p^{-1} in the second bracket introduces a singularity at $p=0$. Further singularities are encountered at the upper and lower limit

because of the first bracket. The occurrence of the singularity at $p=0$ is to be expected according to the discussions carried out in Section VI in conjunction with Eq. (25). One expects that at this point one must take the Cauchy principal value. This is demonstrated in Appendix B. Practically, one can proceed in the following manner.

It is shown in Appendix C that

$$P\int_{-\tilde{a}_2}^{\tilde{a}_1}[(\tilde{a}_1-p)(\tilde{a}_2+p)]^{-1/2}p^{-1}dp=0$$

where P expresses that one has to take the Cauchy principal value. Accordingly, one has

$$Q_a^{mn2}= -a_3\beta^{-n}\int_{-\tilde{a}_2}^{\tilde{a}_1}[(\tilde{a}_1-p)(\tilde{a}_2+p)]^{-1/2}J_n^m(p)dp \tag{73}$$

with

$$J^{mn}(p) = \frac{f^{mn}(p)-f^{mn}(0)}{p}+pg^{(m)}(p^2)\log(1+(1-p^2)^{1/2})+p\bar{g}^{(m)}(p^2) \tag{74}$$

The expression $\frac{f^{mn}(p)-f^{mn}(0)}{p}$ is regular at $p=0$; $f^{mn}(p)$, $f^{mn}(0)$ and the limiting value for $p=0$ of this expression (if it should be needed) are found in Eqs. (57), (58) and (59). The expression $J^{mn}(p)$ is regular throughout the region of integration $-\tilde{a}_2<p<\tilde{a}_1$. Because of the square roots in the bracket of Eq. (73) we set

$$p = \hat{p}\,\frac{\tilde{a}_1-\tilde{a}_2}{2} + \frac{\tilde{a}_1+\tilde{a}_2}{2} \tag{75}$$

One then obtains

$$Q_a^{mn2} = -a_3\beta^{-n}\int_{-1}^{+1}(1-\hat{p}^2)^{-1/2}\ J^{mn}(p)d\hat{p} \tag{76}$$

In $J^{mn}(p)$, one must express p in terms of $\hat{p}$.

Finally, we set

$$\hat{p} = \sin u \tag{77}$$

Then

$$\hat{Q}_a^{mn2} = -a_3\beta^{-n}\int_{-\pi/2}^{\pi/2} J^{mn}(p)du \tag{78}$$

Here p must be regarded as a function of u. The integrand is a periodic function of u. In this case the trapezoidal rule (with equal intervals in u) gives very good results.

In $I_b^m(p)$, one must take the occurrence of the absolute value of p into account.

$$Q_b^{mn2} = a_3\beta^{-n}\pi((m+1)/2)\int_{-\tilde{a}_2}^{\tilde{a}_1}\text{sign } p\ [(\tilde{a}_1-p)(\tilde{a}_2+p)]^{-1/2}f^{mn}(p)dp$$

$$=a_3\beta^{-n}\pi((m+1)/2)[-\int_{-\tilde{a}_2}^{0}[(\tilde{a}_1-p)(\tilde{a}_2+p)]^{-1/2}f^{mn}(p)dp$$

$$+\int_{0}^{\tilde{a}_1}[(\tilde{a}_1-p)(a_2+p)]^{-1/2}f^{mn}(p)dp$$

To remove the singularities for $p = \tilde{a}_1$, and $p = -\tilde{a}_2$ one might apply the transformation used in Q_a^{mn2} again. But because the regions

$-\tilde{a}_2 < p < 0$, and $0 < p < \tilde{a}_1$ are treated separately, it is probably convenient to make separate transformations in the two regions.

Accordingly, we set for the region $-\tilde{a}_2 < p < 0$

$$\tilde{p} = p + \tilde{a}_2 \quad , \quad p = \tilde{p} - \tilde{a}_2 \tag{79}$$

and subsequently $\tilde{p} = u^2$ (80)

and for the region $0 < p < \tilde{a}_1$

$$\tilde{p} = \tilde{a}_1 - p \quad , \quad p = \tilde{a}_1 - \tilde{p} \tag{81}$$

and

$$\tilde{p} = u^2 \tag{82}$$

Then one obtains

$$Q_b^{mn2} = -a_3\beta^{-n}\pi((m+1)/2)\Big[-\int_0^{\tilde{a}_2^{1/2}} (\tilde{a}_1+\tilde{a}_2-u^2)^{-1/2} f^{mn}(u^2-\tilde{a}_2)du$$

$$+ \int_0^{\tilde{a}_1^{1/2}} (\tilde{a}_1+\tilde{a}_2-u^2)^{-1/2} f^{mn}(\tilde{a}_1-u^2)du\Big] \tag{83}$$

Within the regions of integration the integrands are regular functions of u.

The expression

$$Q_c^{mn2} = -a_3\beta^{-n} \int_{-\tilde{a}_2}^{\tilde{a}_1} [(\tilde{a}_1-p)(\tilde{a}_2+p)]^{-1/2} p\, g^{(m)}(p^2) f^{mn}(p)\log|p|dp$$

requires special treatment because of the factor $\log|p|$, although this factor is not detrimental to the convergence of the integral.

A small region $-b < p < +b$ around $p=0$ is, therefore, treated separately.

We write

$$q(p) = [(\tilde{a}_1-p)(\tilde{a}_2+p)]^{-1/2} p\, g^{(m)}(p^2) f^{mn}(p) \qquad (84)$$

Then

$$\tilde{Q}_c^{mn2} = -a_3\beta^{-n}\left[\int_{-\tilde{a}_2}^{-b} q(p)\log|p|dp + \int_{-b}^{+b} q(p)\log|p|dp + \int_{b}^{\tilde{a}_1} q(p)\log|p|dp\right] \qquad (85)$$

In the region $-b < p < b$, the function $q(p)$ is developed with respect to p. Because of the factor p contained in q Eq. (84), there is no constant term in the development:

$$q(p) = b_1 p + b_2 p^2 + b_3\, p^3 + \ldots$$

Consider the integral

$$\int_{-b}^{+b} q(p)\log|p|dp = \int_{-b}^{+b} (b_1 p + b_2 p^2 + b_3 p^3 + \ldots)\log|p|dp$$

The terms with odd powers of p vanish because of the antisymmetry of the integrand. Taking b small enough, one can restrict oneself to terms up to p^3; then only the coefficient b_2 is needed. One finds

$$b_2 = g^m(0)[\tilde{a}_1\tilde{a}_2]^{-1/2}\left[\frac{df^{mn}}{dp}\bigg|_{p=0} + \frac{1}{2}\left(\frac{1}{\tilde{a}_1} - \frac{1}{\tilde{a}_2}\right)f^{mn}(0)\right] \qquad (86)$$

The expressions $f^{mn}(0)$ and $\left.\frac{df^{mn}}{dp}\right|_{p=0}$ are found in Eqs. (58) and (59). Then

$$\int_{-b}^{+b} q(p)\log|p|dp = b_2 \int_{-b}^{+b} p^2 \log|p|dp = (2b_2/3)b^3[(\log b)-(1/3)] \quad (87)$$

To remove the singularity at $p = -\tilde{a}_2$ in the integral for the region $-\tilde{a}_2 < p < b$ we set as above

$$p + \tilde{a}_2 = u^2 \quad , \quad p = u^2 - \tilde{a}_2 \qquad (88)$$

Then

$$\int_{-\tilde{a}_2}^{-b} q(p)\log|p|dp = 2\int_{0}^{(\tilde{a}_2-b)^{1/2}} (\tilde{a}_1+\tilde{a}_2-u^2)^{-1/2} f^{mn}(p) g^{(m)}(p^2) p \log(\tilde{a}_2-u^2)du \qquad (89)$$

Here p is regarded as a function of u. The integrand is regular within the region of integration. One may be concerned of the fact, that b is small and that in the vicinity of the upper limit $u = (\tilde{a}_2-b)^{1/2}$, the argument of log $(\tilde{a}_2-u^2)$ is small, and that, therefore, $|\log(\tilde{a}_2-u^2)|$ increases rapidly. This would make small steps in the numerical integration necessary. A smoother procedure is obtained by a further transformation

$$\log(\tilde{a}_2^{1/2} - u) = -v$$

$$u = \tilde{a}_2^{1/2} - \exp(-v) \qquad (90)$$

$$du = \exp(-v)du$$

The limit of integration $u=0$ becomes $v = -\log(\tilde{a}_2^{1/2})$. The limit of integration $u = (\tilde{a}_2-b)^{1/2}$ becomes $v = -\log[\tilde{a}_2^{1/2} - (\tilde{a}_2-b)^{1/2}]$. Then the integral Eq. (89) is transformed into

$$2\int_{-\log(\tilde{a}_2^{1/2})}^{-\log(\tilde{a}_2^{1/2}-(\tilde{a}_2-b)^{1/2}} (\tilde{a}_1+\tilde{a}_2-u^2)f^{mn}(p)g^{(m)}(p^2)\ p\ [\log(\tilde{a}_2^{1/2}+u]-v]\exp(-v)dv \quad (91)$$

Here p and u are regarded as functions of v. The region of integration would tend to infinity if one allows b to tend to zero The introduction of v makes it possible to carry out the integration with a uniform interval.

The same procedure is applied to the region $b < p < \tilde{a}_1$. One sets

$$\tilde{a}_1 - p = u^2 \quad , \quad p = \tilde{a}_1 - u^2 \qquad (92)$$

This gives $$2\int_0^{(\tilde{a}_1-b)^{1/2}} (\tilde{a}_1+\tilde{a}_2-u^2)^{-1/2}f^{mn}(p)\ g^{(m)}(p^2)p\ \log(\tilde{a}_1-u^2)du$$

The further transformation

$$\log\ (\tilde{a}_1^{1/2}-u) = -v \quad , \quad u = \tilde{a}_1^{1/2} - \exp(-v) \qquad (93)$$

leads to

$$2\int_{-\log(\tilde{a})^{1/2}}^{-\log\ (\tilde{a}_1^{1/2}-b)} (\tilde{a}_1+\tilde{a}_2-u^2)^{1/2}f^{mn}(p)g^{(m)}(p^2)p[\log(a_1^{1/2}+u]-v]\exp(-v)dv \quad (94)$$

Again u and p are regarded as functions of v.

The numerical work proceeds as follows: the shape of the airfoil enters the problem through the value of a_1 and a_2 and through the given upwash i.e., through the value of m and the

function $\hat{w}(y/x)$, which usually has the form (Eq. 34). Moreover, the Mach number M and with it $\beta = (M^2-1)^{1/2}$ are known.

It is assumed that the function routines have been written for

$$f^{mn}(p) \quad , \qquad \text{Eq. (57),}$$

$$f^{mn}(0) \quad , \qquad \text{Eq. (58)}$$

$$(df^{mn}|dp)\Big|_0 , \qquad \text{Eq. (59)}$$

$$g^{(m)}, \text{ and } \bar{g}^{(m)} \quad \text{Eq. (69)}$$

Of course, the latter expressions need to be programmed only for the values of m in which one is interested.

In a collocation method one chooses a number of values of c_k. The number must at least be equal to the number of values n. Initially, one will probably cover the whole range of possible values of c_k, $-a_2 < c_k < a_1$, in order to demonstrate how well the upwash condition can be satisfied by the use of a restricted number of values n.

For these values of c_k one evaluates $\hat{w}^m(c_k)$. These are the inhomogereous terms in the system of equations obtained from Eq. (63).

On the right side a finite number of values of n is admitted. In their choice one will take the symmetry properties of the airfoil into account; for a symmetric planform and a symmetric upwash one will have even values of n.

The further description refers to a specific value of c_k. For this value one determines c, Eq. (66), $\tilde{a}_1$ and $\tilde{a}_2$, Eq. (50), and a_3, Eq. (54). On this basis one finds $\hat{Q}^{mn1}$, Eq. (64). The quantity $\hat{Q}_a^{mn2}$ is defined in Eq. (74). For the pivotal values of u needed for the numerical integration one finds $\hat{p}$, Eq. (77), p Eq. (75) and $T^{mn}(p)$, Eq. (74). The quantity $\hat{Q}_b^{mn2}$ is defined in Eq. (83). With f^{mn} available as a function routine the integrands can

be evaluated immediately. It is likely that one will use different pivotal points for evaluating the two integrals.

Q_c^{mn2} is expressed in Eq. (85) as the sum of three integrals. The second integral is evaluated analytically for some choice of b. With b_2 defined in Eq. (86), the result is given by Eq. (87). The two other integrals are defined in Eqs. (91) and (94). One chooses pivotal values of v and evaluates in the first case u from Eq. (90) and p from Eq. (88). In the second case u is defined in terms of v by Eq. (93) and p by Eq. (92).

One then has Eq. (72)

$$\hat{Q}^{mn2} = Q_a^{mn} + Q_b^{mn} + Q_c^{mn}$$

and

$$\hat{Q}^{mn}_{(c_k)} = \hat{Q}^{mn1} + \hat{Q}^{mn2}.$$

The $\hat{Q}^{mn}(c_k)$ are the matrix elements in the system arising from Eq. (63), n and k are respectively the row and column indices.

After the coefficients b_n are found from this system of equations one can determine h(x,y) from Eqs. (39) and (40).

SECTION VIII

IMPLEMENTATION

In many cases the planform has a tip in which two subsonic edges meet. Then part of the flow can be computed as a conical field, for instance by the procedure shown in Section VII. One determines the coefficients b_n. Then Eq. (35) gives an analytical approximation for the flow field. From these expressions one can compute a pointwise representation of h.

Beyond the region for which the evaluation as a conical field is possible one uses Eq. (32) to determine h at prechosen points of the planform, which will be called control points. We assume that these points lie on lines ξ = const. Since the boundaries of the regions of integrations are Mach lines, it seems reasonable to impose the further condition that the points lie on preselected Mach lines. One thus obtains an arrangement similar to Mach boxes, but without the assumption that within a Mach box the potential is approximated by a constant. The author rather thinks in terms of a pointwise representation of the flow field. The function h at a certain point (x,y) is determined only by data at the planform within its forecone. The computation can therefore be carried out by marching in the x direction. In evaluating the integrals in Eq. (32) it is assumed that the integrations with respect to η are carried out first. Because the computation marches in the x-direction, the integrands in the integration with respect to η are known, except for the immediate vicinity of the point (x,y) under consideration. From the values of h at the control points one can derive an approximation for intermediate points. One might consider an approximation that is piecewise linear in η, but it is probably preferable to use a higher order interpolation because of the denominator $[(x-\xi)^2-\beta^2(y-\eta)^2]^{1/2}$ which becomes zero for $(x-\xi) = \pm\, \beta(\eta-y)$. One has the choice of using information for h that comes only from within the region of integration, or one may involve also

points from outside if one can count on the smoothness of h. The first possibility may be simpler.

We have seen, that in the vicinity of the point (x,y) the integral over η does not vanish, although at the point (x,y) the upper and lower limit of the integral coincide. A limiting value $h_{\eta\eta} = h^{(22)}$ is encountered for $\eta \to y$. But at the station (x,y) and the station upstream of it this limiting value is not available (unless one admits information from outside the region of influence). Here the following procedure is suggested. We assume that in the entire triangle shown in Figure 5, h is given by one analytical expression.

The starting point is Eq. (31). Developing the integrand with respect to $\bar{\eta}$ one obtains

$$\beta^{-1}\iint \frac{\bar{\eta}^2 h^{(22)}(\xi,y)\,d\xi\,d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}} = \beta^{-1}\int h^{22}(\xi,y)\left(\int_{-1}^{+1}\frac{\bar{\eta}^2\,d\bar{\eta}}{(1-\eta^2)^{1/2}}\right)d\xi$$

$$= \bar{\beta}^{1}\ (\pi/2)\int h^{(22)}(\xi,y)\,d\xi$$

Along the line 2,0,1 one has

$$h_{\eta\eta} = \frac{h_1+h_2-2h_o}{\Delta\eta^2}$$

where $\Delta\eta$ is the interval in η. In the evaluation of the flow field one obtains in an intermediate step h_ξ. Therefore, h_ξ is available at points 2,0,1 and one has also

$$h_{\eta\eta\xi} = \frac{h_{1\xi}+h_{2\xi}-2h_{o}\xi}{\Delta\eta^2}$$

For the extrapolation one therefore can use

$$h_{22}(y,\xi) = \frac{h_1+h_2-2ho}{\Delta\eta^2} + (\xi-\xi_o)\ \frac{h_{1\xi}+h_{2\xi}-2h_{o\xi}}{\Delta\eta^2}$$

Carrying out the integration from $\xi = \xi_o$ to $\xi = \xi_s = \xi_o + \beta\Delta\eta$ one then obtains for the contribution of the triangular region to the integral

$$\pi/2 \quad [(h_1+h_2-2h_o)/\Delta\eta) + (\beta/2)(h_{1\xi}+h_{2\xi}-2h_{o\xi})]$$

where $\Delta\eta$ is the interval in η.

Furthermore

$$h_{\eta\eta\xi} = \frac{h_{\xi 1}+h_{\xi 2}-h_{\xi o}}{\Delta\eta^2}$$

This is used for an extrapolation

$$h^{(22}(\xi) = \frac{h_1+h_2-2h_o}{\Delta\eta^2} + \xi \, \frac{h_{\xi 1}+h_{\xi 2}-2_{\xi o}}{\Delta\eta^2}$$

The interval of integration extends from $\xi=0$ (at points 0,12) to $\xi = \beta\Delta\eta$ at point 5.

The contribution of the entire triangle is therefore given by

$$\pi/2 \, \frac{\beta}{\Delta\eta} \, [h_1+h_2-2h_o + \frac{\beta}{2} \Delta\eta(h_{1\xi}+h_{2\xi}-2h_{o\xi})]$$

APPENDIX A

DISCUSSION OF DERIVATIVES OF $\phi^{(s)}$ WITH RESPECT TO z WITHIN AREA II

The area II is defined in the main text. It is bounded by the leading edge, by the straight line $\xi = \xi_1$ (where ξ_1 is chosen so that this line intersects the hyperbola $(1-\xi)^2-(\eta-y)^2 - z^2 = 0$ twice within the planform) and by two portions of this hyperbola. One notices that the variable z with respect to which the differentiations are carried out enters the expression ϕ^{sII} only in the form of the parameter $s = z^2$.

We write

$$\phi^{(sII)}(x,y,z) = \hat{\phi}(x,y,s)$$

Then

$$\phi^{(dII)}(x,y,z) = 2z\hat{\phi}_s(x,y,s)$$

$$\phi_z^{(dII)}(x,y,z) = 2\hat{\phi}_s(x,y,s) + 4z^2\hat{\phi}_{ss}(x,y,s)$$

In the limit z = 0 one obtains

$$\phi^{(d,II)}(x,y,z) = 0$$

$$\phi_z^{(d,II)}(x,y,z) = \phi_s(x,y,s)$$

provided that ϕ_{ss} exists. Whether ϕ_{ss} exists in the limit $z = 0$ is not immediately obvious, this is the reason for the discussions carried out in this Appendix. But after the existence of ϕ_{ss} has been established, it is no longer necessary to evaluate it in detail, because we are only interested in the limit $z \to 0$.

The leading edge is given by the equation

$$\eta = g(\xi) \tag{A.1}$$

The inverse of g is denoted by f

$$\xi = f(\eta) \qquad (A.2)$$

We have introduced

$$\eta = y + \bar{\eta}\,[(x-\xi)^2 - s]^{1/2}$$

$$\bar{\eta} = (\eta - y)[(x-\xi)^2 - s]^{-1/2}$$

In the $\xi\bar{\eta}$-plane the leading edge is then given by

$$\bar{\eta} = \bar{g}(\xi, s, x, y) \qquad (A.3)$$

with

$$\bar{g}(\xi, s, x, y) = (g(\xi) - y)[(x-\xi)^2 - s]^{-1/2} \qquad (A.4)$$

In the present context the arguments x and y are kept constant. In many cases, they will therefore not be listed as arguments in the functions in question. The inverse of the function $\bar{g}$ at constant s, x, and y is denoted by $\bar{f}(\bar{\eta}, s, x, y)$. Accordingly, we have as alternative to Eq. (A.3) for the equation of the leading edge in the $\xi\bar{\eta}$-system

$$\xi = \bar{f}(\bar{\eta}, s, x, y) \qquad (A.5)$$

For subsonic leading edges one has $|dg/d\xi| < \beta^{-1}$. For simplicity, the present discussions will be carried out for $M^2 = 2$. Then $\beta = 1$. Depending upon the orientation of the portion of the leading edge under consideration the function g is either monotonically increasing or decreasing. For $\bar{g}$, however, this is not necessarily correct. In Figure 6, the curve AB is a leading edge along which η decreases monotonically with ξ; yet it is tangent to a straight line through the point (x,y). Such a straight line is the limit of some curve $\bar{\eta}$ = const as $z \to 0$. The

variable $\bar{\eta}$ therefore first increases then decreases as one travels from A to B along the leading edge. If this should occur the area is denoted by IIb; in the ξ-directive it extends from $\xi=\xi_2$ to $\xi=\xi_2'$.

Let us first consider the problem with subsonic leading edges. Figure 7 shows such a configuration in the $\xi\eta$ - and in the $\xi\bar{\eta}$-planes. At the leading edge the potential and with it h have a square root singularity. The detailed discussions carried out here, are made necessary because the leading edge moves in the $\xi\bar{\eta}$ system as s varies. We denote by y the local sweep angle and introduce a local Cartesian coordinate system uv, where the u-direction is perpendicular to the leading edge. Let ξ^1, η^1 be the coordinates of the point at the leading edge under consideration. Then (Fig. 8)

$$\xi-\xi^1 = u\cos\gamma + v\sin\gamma \qquad u = (\xi-\xi^1)\cos\gamma - (\eta-\eta^1)\sin\gamma$$

$$\eta-\eta^1 = -u\sin\gamma + v\cos\gamma \qquad v = (\xi-\xi^1)\sin\gamma + (\eta-\eta^1)\cos\gamma$$

Because of the square root singularity one has locally

$$h = \text{const}\ u^{1/2} = \text{const}[(\xi-\xi^1)\cos\gamma - (\eta-\eta^1)\sin\gamma]^{1/2}$$

Therefore for $\eta = \eta^1 = \text{const}$

$$h = \text{const}(\cos\gamma)^{1/2}(\xi-\xi^1)^{1/2}$$

used for $\quad \xi = \xi^1 = \text{const}$

$$h = \text{const}\ (\sin\gamma)^{1/2}\ (\eta^1-\eta)^{1/2}$$

The flow field in the vicinity of the intersection of the two subsonic leading edges is rather complicated. The basic structure is that of a conical field, but higher order corrections occur if the leading edges are not straight or if the

upwash is not constant. This vicinity is best considered in the original $\xi\eta$-system. This is the region IIa in Figure 9. In order to avoid dealing at the same time with two boundaries that depend upon s, the remainder of the region is subdivided by a line $\bar{\eta} = 0$. One thus obtains the regions IIc and IId. In the vicinity of the point where a line $\bar{\eta} =$ const is tangent to the leading edge, if such a point should occur, we introduce for this vicinity a separate region IIb.

In the tip region IIa, the boundaries do not depend upon s. One has

$$\phi^{(s,IIa)}(x,y,z) = \iint \frac{h(\xi,\eta)d\xi d\eta}{[(x-\xi)^2-(\eta-y)^2-s]^{1/2}}$$

$$\phi^{(d,IIa)}(x,y,z) = z\iint \frac{h(\xi,\eta)d\xi d\eta}{[(x-\xi)^2-(\eta-y)^2-s]^{3/2}}$$

The denominator is always different from zero. Hence

$$\phi^{(d,IIa)}(x,y,0) = 0$$

$$\phi^{(d,IIa)}(x,y,0) = \iint \frac{h(\xi,\eta)d\xi d\eta}{[(x-\xi)^2-(\eta-y)^2]^{3/2}} \qquad \text{(A.6)}$$

To bring the expression into a form which will be encountered in other regions, we introduce $\bar{\eta}$. Specializing immediately to $s = z^2 = 0$ one has $\eta = y + \bar{\eta}(x-\xi)$.

Then

$$\phi_z^{(d,IIa)}(x,y,0) = \iint \frac{h(\xi,y+\bar{\eta}(x-\xi))d\xi d\bar{\eta}}{(x-\xi)^2(1-\bar{\eta}^2)^{3/2}}$$

Now

$$\frac{d}{d\bar{\eta}}\left(\frac{\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}}\right) = \frac{1}{(1-\bar{\eta}^2)^{3/2}}$$

In the last equation we carry out the integration with respect to $\bar{\eta}$ first, and transform the inner integral by an integration by parts. At the upper and lower limits (the intersections of a line ξ = const with the leading edges) h vanishes and one obtains

$$\phi_z^{(d,IIa)}(x,y,0) = -\int\frac{1}{x-\xi}\left(\int\frac{h^{(2)}(\xi,y+\bar{\eta}(x-\xi))\bar{\eta}d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}}\right)d\xi \quad (A.7)$$

The function $h^{(2)}$ behaves at the limits as $\Delta\bar{\eta}^{-1/2}$; the inner integral converges. The denominator does not vanish since $\bar{\eta} < 1$. Returning to the original coordinates one obtains an integrand familiar from Eq. (32)

$$\phi_z^{(d,IIa)}(x,y,0) = -\iint\limits_{AIIa}\frac{(\eta-y)h^{(2)}(\xi,\eta)}{(x-\xi)^2[(x-\xi)^2-(\eta-y)^2]^{1/2}}d\xi d\eta \quad (A.8)$$

One remembers that in this Appendix $\beta^2 = 1$. In a region IIb (if it should occur) the integration with respect to $\bar{\eta}$ is carried out first. We write

$$\phi^{(s)} = \int_{\xi_2}^{\xi_2'} F(\xi,s)d\xi \quad (A.9)$$

where

$$F(\xi,s) = \int_0^{\bar{\eta}=\bar{g}(\xi,s)}\frac{h(\xi,y+\bar{\eta}q(\xi,x,s))}{(1-\bar{\eta}^2)^{1/2}}d\bar{\eta}$$

$$q(\xi,x,s) = ((x-\xi)^2-s)^{1/2}$$

$$s = z^2$$

At the upper limit (which is a point of the leading edge) h has a square root singularity

$$h \sim \text{const}\ (g(\xi)-\eta)^{1/2}$$

This expression is rewritten in terms of $\bar{\eta}$ and $\bar{g}$, Eqs. (13) and (17)

$$h \sim \text{const}\ [g(\xi)-y-\bar{\eta}\cdot q]^{1/2}$$

$$h \sim \text{const}\ q\ [\bar{g}(\xi)-\eta)]^{1/2}$$

The factor $q(\xi,s)$ has no singularity in the whole of the region II. The discussions of Appendix E show, that $\partial^2 F/\partial s^2$ is bounded. One therefore finds

$$\phi_{zz}^{(s,IIb)} = \phi_z^{(d,IIb)} = 2\int_{\xi_2}^{\xi_2'} (\partial F/\partial s)\Big|_{s=0} d\xi$$

Here

$$\partial F/\partial s\Big|_{s=0} = -\ (1/2)\int_0^{\bar{\eta}=\bar{g}(\xi,0)} \frac{\bar{\eta}h^{(2)}(\xi,(y+\bar{\eta}(x-\xi))d\bar{\eta}}{(x-\xi))1-\bar{\eta}^2)^{1/2}}$$

Returning to the original coordinates ξ,η, one obtains the integrand occurring in Eq. (32)

The regions IIc (or IId) are subdivided into regions IIc1 and IIc2 by the line $\bar{\eta}=\bar{g}(\xi_2,s)$, Figure 11. This boundary moves as s changes. Consider

$$\phi^{(IIc1)}(x,y,z) = \int_{\bar{\eta}=\bar{g}(\xi_2,s)}^{1} \frac{F(\bar{\eta},s)d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}} \qquad (A.10)$$

where

$$F(\bar{\eta},s) = \int_{\bar{f}(\bar{\eta},s)}^{\xi_1} h(\xi,(y+\bar{\eta}q(\xi,x,s))d\xi \qquad (A.11)$$

As before $\bar{f}(\bar{\eta},s)$ denotes the inverse of the function $\bar{\eta}=\bar{g}(\xi,x,y,s)$ at constant x,y,s. The lower limit in F lies at a point of the leading edge. There the function h has a square root singularity. We write

$$h(\xi,y+\bar{\eta}\bar{q})^{1/2} = [\xi-\bar{f}(\bar{\eta},s)]^{1/2}\bar{h}(\xi,\bar{\eta},s)$$

(The dependence upon x and y does not appear in the arguments because they are kept constant.) Here $\bar{h}$ is free of singularities. Now the discussions of Appendix E are again applicable. The first and second derivative of F with respect to s are bounded, and as the second derivative wit- respect to z is needed for z = 0. Only, there is no need to evaluate F_{ss}. F_s is obtained from Eq. (A.11). At the lower limit the function h vanishes. The discussions of Appendix E show, that (for the first derivative) the square root singularity encountered at this point does not matter.

$$F_s = -(1/2)\int_{\bar{f}(\bar{\eta},s)}^{\xi_1} \frac{\bar{\eta}h^{(2)}(\xi,y+\bar{\eta}q(\xi,x,s))d\bar{\eta}}{q} \qquad (A.12)$$

Then, by differentiating Eq. (A.10) with respect to s and substituting Eq. (A.12).

$$\phi_{s(x,y,0)}^{(s,IIb)} = -(\partial\bar{g}/\partial s)\Big|_{\substack{\xi=\xi_2\\ s=0}} \frac{F(\bar{g}(\xi_2,0),0)}{(1-\bar{g}(\xi_2,0)^2)^{1/2}}$$

$$-\frac{1}{2}\int_{\bar{\eta}=\bar{g}(\xi_2,0)}^{1} \frac{\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}} \left(\int_{\bar{f}(\bar{\eta},0)}^{\xi_1} \frac{h^{(2)}(\xi,y+\bar{\eta}(x-\xi))d\xi}{x-\xi}\right)d\bar{\eta} \qquad (A.13)$$

Going back to the original coordinates ξ and η, one obtains again an expression with the integrand of Eq. (32), except for the first term on the right, which arises because of the s-dependence of the lower limit in Eq. (A.10)

In the region IIc one has

$$\phi^{(s,IIc)}(x,y,z) = \int_{0}^{\bar{\eta}=\bar{g}(\xi_2,s)} \frac{\tilde{F}(\bar{\eta},s)d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}} \qquad (A.14)$$

with

$$\tilde{F}(\bar{\eta},s) = \int_{\xi_2}^{\xi_1} h(\xi,(y+\bar{\eta}q(\xi,x,s))d\xi \qquad (A.15)$$

$$= \int_{\xi_2}^{\xi_1} (\xi-\bar{f}(\bar{\eta},s))^{1/2}\tilde{h}(\xi,\bar{\eta},s)d\xi$$

A comparison of Eq. (A.15) and (A.11) shows, that

$$F(\bar{g}(\xi_2,s),s) = \tilde{F}(\bar{g}(\xi_2,s),s) \qquad (A.16)$$

The limits in the function $\tilde{F}$ do not depend upon s. A complication arises, however, because for $\bar{\eta}=\bar{g}(\xi_2,s)$ the integrand

has a point in common with the leading edge. One will surmise that this is unessential, but some discussion is desirable. One obtains from Eq. (A.15)

$$\tilde{F}_s(\eta,s)=\int_{\xi_2}^{\xi_1}(-1/2)[\bar{f}^{(2)}(\bar{\eta},s)(\xi-\bar{f}(\bar{\eta},s))^{-1/2}h(\xi,\bar{\eta},s)$$

$$+(\xi-\bar{f}(\bar{\eta},s))^{1/2}\tilde{h}^{(3)}(\xi,\bar{\eta},s)]d\xi \qquad \text{(A.17)}$$

The first term in the integrand tends to infinity for the lower limit $\xi=\xi_2$ if $\bar{\eta}=\bar{g}(\xi_2,s)$, that is at the upper limit in Eq. (A.13) but the integral converges. In addition, one must show, that F_{ss} is bounded (see Appendix E). One has

$$\tilde{F}_{ss}(\bar{\eta},s) = \int_{\xi_2}^{\xi_1}\{[-(1/4)(\bar{f}^{(2)}(\eta,s))^2(\xi-\bar{f}(\eta,s))^{-3/2}$$

$$-(1/2)\bar{f}^{(2,2)}(\xi-\bar{f}(\bar{\eta},s))^{-1/2}]\tilde{h}(\xi,\bar{\eta},s)$$

$$-\bar{f}^{(2)}(\bar{\eta},s)(\xi-\bar{f}(\bar{\eta},s)^{-1/2}h^{(3)}(\xi,\bar{\eta},s)+(\xi-\bar{f}(\bar{\eta},s)^{1/2}\tilde{h}^{(33}(\xi,\bar{\eta},s)\}d\xi$$

After the integration has been carried out the term with the factor $(\xi-\bar{f}(\bar{\eta}s))^{-3/2}$ gives a result which tends to infinity as $\bar{\eta}$ approaches $\bar{g}(\xi_2,s)$. This contribution is

$$O(\xi_2-f(\eta,s))^{-1/2}$$

Let $\qquad \bar{\eta}=\bar{g}(\xi_2,s)-\varepsilon \qquad$ (A.18)

Then $\qquad \bar{f}(\bar{\eta},s) = \bar{f}(\bar{g}(\xi_2,s)-\varepsilon,s)$

But $\bar{f}$ is the inverse of $\bar{g}$, therefore

$$\bar{f}(\bar{g}(\xi_2,s),s) = \xi_2$$

and $\bar{f}(\bar{g}_2,s)-\varepsilon,s) = \xi_2 - O(\varepsilon)$

The critical term

$$(\xi_2-\bar{f}(\bar{\eta},s)^{-1/2} \text{ is therefore } O(\varepsilon^{-1/2}).$$

If one forms $\phi_{ss}^{(s,IIc)}$ from Eq. (A.13) one obtains in the vicinity of $\bar{\eta}=\bar{g}(\xi_2,s)$.

$$\int O(\varepsilon^{-1/2})d\varepsilon = O(\varepsilon^{1/2})$$

With ε defined by Eq. (A.18) the contribution of F_{ss} to ϕ_{ss} is therefore bounded. It suffices, therefore, if one evaluates only $\phi_s^{(s,IIc)}$.

$$\phi_s^{(s,IIc)} = (\partial\bar{g}/\partial s)\Big|_{\substack{\xi=\xi_2\\ s=0}}(1-\bar{g}(\xi_2,0)^2)^{-1/2}\tilde{F}(\bar{g}(\xi_2,0),0)$$

$$+\int_0^{\bar{\eta}=g(\xi_2,0)} \frac{\tilde{F}^{(2)}(\bar{\eta},0)d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}} \qquad (A.19)$$

Because of Eq. (A.16) the first term on the right in this expression cancels the first term on the right in Eq. (A.13). In the second term in Eq. (A.19), $\tilde{F}^{(2)} = \tilde{F}_s$ is substituted from Eq. (A.17), furthermore, h (instead of $\bar{h}$) is reintroduced. Returning to the coordinates ξ and η one arrives again at the integrand in Eq. (32).

If the leading edge is partially subsonic and partially supersonic one can distinguish on the wing surface between the regions A and B (Fig. 12). They are separated by the Mach wave emanating from the corner between the subsonic and supersonic leading portions of the leading edge. The boundary of the region of integration for a given point (x,y,z) always consists of the hyperbola and part of the leading edge. If the point (x,y) lies in the region A, then the part formed by the leading edge is

entirely supersonic. At the leading edge the potential and with it h are zero, as always, and h increases linearly with the distance from the leading edge. The precautions which we took because of the square root singularity at a subsonic leading edge and because of the conical field at a tip formed by two subsonic leading edges are not needed, and one arrives without the detailed discussions shown for such cases in this appendix at Eq. (32). If no points other than those of region A were considered, it would be preferable to represent the flow field by a source distribution. The function h is then directly determined by the given upwash Eq. (23) and the potential is found by integrations, for instance from Eq. (18). The flow field in the region A is not affected by the conical field which arises at the juncture of the supersonic and subsonic leading edges.

The cases where the point (x,y) lies within the region B differs from those with two subsonic edges by the treatment of the vicinity of the juncture of the two edges, Figure 13. We distinguish between the subregion CDE, lying downstream of the Mach wave CD (Region IIa1), and the subregion CDFG, downstream of the supersonic part of the leading edge, but upstream of the Mach wave CD (Region IIa2).

The character of the flow field in the region IIa1 is similar to that of the region formerly denoted by IIa. The contribution of the region IIa1 to the upwash at the point (x,y) is again evaluated in the original $\xi\eta$-system, and one obtains as an intermediate result Eq. (A.6). In a further step performed to bring this result into the form which arises in other regions, we introduced $\bar{\eta}$ instead of η and carried out an integration by parts Eq. (A.7). Subsequently, we returned to the original $\xi\eta$ system Eq. (A.8). This procedure is applied again. But here the lower limit for η is not a leading edge where h is zero, but the Mach wave CD, where the potential is different from zero.

One obtains

$$\phi_z^{(dIIa)} = -\int_{\xi_C}^{\xi_D}\left(\frac{h(\xi,\eta)(\eta-y)}{(x-\xi)^2[(x-\xi)^2-(\eta-y)^2]^{1/2}}\right)\Bigg|_{\eta=\eta-(\xi-\xi_C)} d\xi \; - \iint_{IIA1}\left(\frac{h^{(2)}(\xi,\eta)(\eta-y)d\xi d\eta}{(x-\xi)^2[(x-\xi)^2-(\eta-y)^2]^{1/2}}\right) \tag{A20}$$

The first term on the right is the contribution of the Mach wave CD. The second term has the form familiar from previous discussions. In the first term we introduce η instead of ξ as variable of integration

$$\eta = \eta_C - (\xi-\xi_C)$$

One remembers, that for Mach waves $d\xi/d\eta=\pm 1$, because here $\beta=1$. Then the integral assumes the form

$$\int_{\eta_C}^{\eta_D}\left(\frac{h(\xi,\eta)(\eta-y)}{(x-\xi)^2[(x-\xi)^2-(\eta-y)^2]^{1/2}}\right)\Bigg|_{\xi=\xi_C-(\eta-\eta_C)} d\eta \tag{A.21}$$

We shall show that this term will be cancelled, provided that one expresses the solution in the region IIa2 by a doublet distribution, and evaluates the upwash at the point (x,y) accordingly. It is true, the determination of the solution in region A is simpler if one applies a source distribution, the use of the doublets has the advantage that one obtains in all case the same expressions in the integral equation.

Because part of the boundary of the region IIa2 is formed by the hyperbola we introduce again $\bar{\eta}$ and perform the integration with respect to ξ first. Then

$$\phi^{(s,IIa2)} = \int_{\bar{\eta}=-1}^{\bar{\eta}_C} \frac{F(\eta,s)}{(1-\bar{\eta}^2)^{1/2}}\, d\bar{\eta} \tag{A.22}$$

with

$$F(\bar{\eta},s) = \int_{\bar{f}(\bar{\eta},s)}^{\xi_{upper}(\bar{\eta})} h(\xi,y+\bar{\eta}q)\, d\xi$$

Here, since the discussions are carried out for $\beta=1$

$$q = [(x-\xi)^2-s]^{1/2} \qquad , \qquad s=z^2.$$

For $\bar{\eta}=\bar{\eta}_C$ the two limits of the last integral coincide. Therefore,

$$F(\bar{\eta}_c,s) = 0$$

The function $\xi=\bar{f}(\bar{\eta},s)$ is the equation of the (supersonic) leading edge in the $\xi\bar{\eta}$ system.

The upper limit ξ_{upper} is partially given by the line DF, and partially by the line CD.

Differentiating Eq. (A.22) with respect to z, one obtains

$$\phi_z^{(d,IIa2)} = 2 \int_{\bar{\eta}=-1}^{\bar{\eta}_A} \frac{F^{(2)}(\bar{\eta},s)\, d\bar{\xi}}{(1-\bar{\eta}^2)^{1/2}} \tag{A.23}$$

Now

$$F^{(2)}(\bar{\eta},s) = \frac{d\xi_{upper}}{ds}\, h(\xi,y+\bar{\eta}q)\Big|_{\xi=\xi_{upper}} \tag{A.24}$$

$$-(1/2)\int_{\bar{f}(\bar{\eta},s)}^{\xi_{upper}(\bar{\eta})} \frac{\bar{\eta}h^{(2)}(\xi,y+\bar{\eta}q)d\xi}{q}$$

The last term, evaluated for s=0 and substituted into Eq. (A.23) gives a contribution

$$-\iint_{\Pi D} \frac{\bar{\eta}h^{(2)}(\xi,y+\bar{\eta}(x-\xi))}{(1-\bar{\eta}^2)^{1/2}(x-\xi)}\, d\xi d\bar{\eta}$$

In terms of the original coordinates $\xi\eta$ it assumes the form familiar from Eq. (32).

In the first term of Eq. (A.24) $d\xi_{upper}/ds=0$ along DF.

Since the slope of CD for $\beta=1$ is -1, one has in the $\xi\eta$ plane

$$\xi_{upper} - \xi_c + \eta - \eta_c = 0$$

or since $\xi_c = x_c$, $\eta_c = y_c$

$$-(x-\xi_{upper}) + (x-x_c) + (\eta-y_c) = 0 \qquad (A.25)$$

Here η is expressed by $\bar{\eta}$

$$\eta = y + \bar{\eta}q$$

Then Eq. (A.25) for ξ_{upper} becomes, after substitution of q

$$-(x-\xi_{upper}) + (x-\xi_c) + (y-y_c) + \bar{\eta}[(x-\xi_{upper})^2-s]^{1/2}=0 \qquad (A.26)$$

We note that one obtains for s=0

$$-(x-\xi_{upper})(1-\bar{\eta}) + (x-x_c) + (y-y_c) = 0 \qquad (A.27)$$

To obtain $d\xi_{upper}/ds$ we differentiate Eq. (A.26) at constant $\bar{\eta}$ with respect to s

$$d\xi_{upper} + \bar{\eta}\,\frac{-(x-\xi_{upper})d\xi_{upper}-(1/2)ds}{[(x-\xi_{upper})^2-s]^{1/2}}$$

and after specialization to s=0

$$d\xi_{upper}/ds = \frac{1}{2}\,\frac{\bar{\eta}}{(x-\xi_{upper})(1-\bar{\eta})} \qquad (A.28)$$

This is now substituted into the first term of Eq. (A.24) and subsequently into Eq. (A.23).

Then one obtains

$$\int_{\bar{\eta}_D}^{\bar{\eta}_C} \frac{h(\xi_{upper}(\bar{\eta}), y+\bar{\eta}q))\bar{\eta}d\bar{\eta}}{(x-\xi_{upper}(\bar{\eta}))(1-\bar{\eta})}$$

To compare this expression it with Eq. (A.21), one must replace the variable $\bar{\eta}$ by η. One has from Eq. (A.14.3)

$$d\xi_{upper}(1-\bar{\eta}) + (x-\xi_{upper})d\bar{\eta}=0$$

Along CD, $d\xi_{upper} = -d\eta$. Therefore

$$\frac{d\bar{\eta}}{d\eta} = \frac{(1-\bar{\eta})}{x-\xi_{upper}}$$

Thus the above expression assumes the form

$$\int_{\eta_D}^{\eta_C} \left. \frac{h(\xi,\eta)(\eta-y)}{(x-\xi)^2[(x-\xi)^2-(\eta-y)^2]^{1/2}} \right|_{\xi=\xi_C-(\eta-\eta_C)} d\eta$$

Because the limits of integration are interchanged, this is, indeed, the negative of the expression (A.21). In the combined

contributions of the regions IIa1 and IIa2, the integrals along the line CD cancel.

APPENDIX B

EVALUATION OF CERTAIN INTEGRALS

The integral $I^m(p)$ is introduced in Eq. (68)

$$I^m(p) = \int_0^{(1+|p|)^{-1}} \frac{\hat{\xi}^{m+2} d\hat{\xi}}{(1-\hat{\xi})^2[(1-\hat{\xi})^2-p^2\hat{\xi}]^{1/2}} \tag{B.1}$$

The radicand is rewritten

$$(1-\hat{\xi})^2 - p^2\hat{\xi}^2 = (1-p^2)^{-1}[(\hat{\xi}(1-p^2)-1)^2-p^2]$$

We introduce

$$\hat{\xi}(1-p^2)-1 = -q \tag{B.2}$$

Then

$$\hat{\xi} = \frac{1-q}{1-p^2}$$

$$1 - \hat{\xi} = \frac{q-p^2}{1-p^2}$$

$$d\hat{\xi} = \frac{-dq}{1-p^2}$$

The lower limit of $I^m(p)$ transforms into $q=1$, the upper limit into $q = |p|$. Then one obtains

$$I^m(p) = (1-p^2)^{-m-(1/2} \int_{|p|}^{1} \frac{(1-q)^{m+2}dq}{(q-p^2)^2[q^2-p^2]^{1/2}}$$

One has

$$(1-q)^{m+2} = [(1-p^2) - (q-p^2)]^{m+2}$$

$$=(1-p^2)^{m+2} - \binom{m+2}{1}(1-p^2)^{m+1}(q-p^2) + \binom{m+2}{2}(1-p^2)^{m-2}(q-p^2)^2$$

In particular for m=2

$$\frac{(1-q)^2}{(q-p^2)^2} = \frac{(1-p^2)^2}{(q-p^2)^2} - \frac{2(1-p^2)}{(q-p^2)} + 1,$$

for m=1

$$\frac{(1-q)^3}{(q-p^2)^2} = \frac{(1-p^2)^3}{(q-p^2)^2} - \frac{3(1-p^2)^2}{(q-p^2)} + (3-2p^2) - q$$

for m=2

$$\frac{(1-q)^4}{(q-p^2)^2} = \frac{(1-p^2)^4}{(q-p^2)^2} - \frac{4(1-p^2)^3}{(q-p^2)} + (6-8p^2+3p^4) + (-4+2p^2)q + q^2$$

These values of m are probably sufficient for practical purposes; expression for greater values of m are easily derived when needed. We introduce the following integrals

$$\tilde{I}_{-2} \int_{|p|}^{1} \frac{dq}{(q-p^2)^2(q^2-p^2)^{1/2}}$$

$$\tilde{I}_{-1} \int_{|p|}^{1} \frac{dq}{(q-p^2)(q^2-p^2)^{1/2}}$$

$$\tilde{I}_o = \bar{I}_o = \int_{|p|}^{1} \frac{dq}{(q^2-p^2)^{1/2}}$$

$$\bar{I}_n = \int_{|p|}^{1} \frac{q^n dq}{(q^2-p^2)^{1/2}}$$

Then

$$I^0(p) = (1-p^2)^{3/2}\ \tilde{I}_{-2} - 2(1-p^2)^{1/2}\ \tilde{I}_{-1} + (1-p^2)^{-1/2}\bar{I}_o \qquad \text{(B.3)}$$

$$I^1(p) = (1-p^2)^{3/2}\ \tilde{I}_{-2} - 3(1-p^2)^{1/2}\ \tilde{I}_{-1} + (1-p^2)^{-3/2}(3-2p^2)I_o$$
$$- (1-p^2)^{-3/2}\bar{I}_1$$

$$I^2(p) = (1-p^2)^{3/2}\ \tilde{I}_{-2} - 4(1-p^2)^{1/2}\ \tilde{I}_{-1} + (1-p^2)^{-5/2}(6-8p^2+3p^4)I_o$$
$$+ (1-p^2)^{-5/2}(-4+2p^2)\bar{I}_1 + (1-p^2)^{-5/2}\bar{I}_2$$

The integrals $\bar{I}_n$ (in their indefinite form) are connected by the recurrence relation

$$\bar{I}_n = (1/n)[q^{n-1}(q^2-p^2)^{1/2} + (n-1)p^2\bar{I}_{n-2}] \qquad n > 2$$

This is shown as follows

$$\bar{I}_n = \int\frac{q^n dq}{(q^2-p^2)^{1/2}} = \int\frac{q^{n-1} q dq}{(q^2-p^2)^{1/2}}$$
$$= q^{n-1}(q^2-p^2)^{1/2} - (n-1)\int q^{n-2}(q^2-p^2)^{1/2}dq$$
$$= q^{n-1}(q^2-p^2)^{1/2} - (n-1)\int\frac{q^n dq}{(q^2-p^2)^{1/2}} + (n-1)p^2\int\frac{q^{n-2}dq}{(q^2-p^2)^{1/2}}$$

The two integrals on the right are respectively $\bar{I}_n$ and $\bar{I}_{n-2}$. This leads to the above recurrence relation. One verifies by differentiation

$$\bar{I}_o = \int_{|p|}^{1} \frac{dq}{(q^2-p^2)^{1/2}} = \log [q + (q^2-p^2)^{1/2}] \Big|_{|p|}^{1} = \log \frac{1+(1-p^2)^{1/2}}{|p|}$$

$$\bar{I}_1 = \int_{|p|}^{1} \frac{qdq}{(q^2-p^2)^{1/2}} = (q^2-p^2)^{1/2} \Big|_{|p|}^{1} = (1-p^2)^{1/2}$$

Then, according to the above recurrence relation

$$\bar{I}_2 = (1/2)\ (1-p^2)^{1/2} + (p^2/2)\bar{I}_o$$

The integrals $\tilde{I}_{-2}$ and $\tilde{I}_{-1}$ can be found in tables. The author has used Ref. 1, Formula 234.3a for $\tilde{I}_{-2}$ and, with some modification, formula 234.3b for $\tilde{I}_{-1}$. One has

$$\int \frac{dq}{(q-\alpha)(q^2-a^2)^{1/2}} = (a^2-\alpha^2)^{-1/2} \text{ arc cos } \frac{a^2-aq}{a(q-a)}$$

$$\int \frac{dq}{(q-\alpha)^2(q^2-a^2)^{1/2}} = (a^2-\alpha^2)^{-1}[\frac{(q^2-a^2)^{1/2}}{q-\alpha} + \alpha \int \frac{dq}{(q-d)(q^2-a^2)^{1/2}}$$

At present $a=p$, $\alpha=p^2$, then $a^2-\alpha^2 = p^2(1-p^2)$

Therefore

$$\tilde{I}_{-1}=|p|^{-1}(1-p^2)^{1/2} \text{ arc cos } \frac{p(1-q)}{q-p^2} \Big|_{q=|p|}^{1} = |p|^{-1}(1-p^2)^{-1/2}\pi/2$$

$$\tilde{I}_{-2}= p^{-2}(1-p^2)^{-1}\ \frac{(q^2-p^2)^{1/2}}{q-p^2} \Big|_{q=|p|}^{1} + p^2\ \tilde{I}_{-1} \qquad \text{(B.4)}$$

$$\tilde{I}_{-2} = \frac{1}{p^2(1-p^2)^{3/2}}\ [1+|p|\pi/2]$$

Substituting these expressions into Eq. (B.3), one obtains

$$I^{o}(p)=\frac{1}{p^2}-\frac{1}{|p|}\frac{\pi}{2}+(1-p^2)^{-1/2}\,[\log(1+(1-p^2)^{1/2})-\log|p|]$$

$$I^{1}(p)=\frac{1}{p^2}-\frac{1}{|p|}\pi+(1-p^2)^{-3/2}(3-2p^2)[\log(1+(1-p^2)^{1/2})-\log|p|]-(1-p^2)^{-1}$$

$$I^{2}(p)=\frac{1}{p^2}-\frac{1}{|p|}\frac{3\pi}{2}+(1-p^2)^{-2}(-(7/2)+2p^2)+$$
$$+(1-p^2)^{-5/2}(6-(15/2)p^2+3p^4)[\log(1+(1-p^2)^{1/2})-\log|p|]$$

APPENDIX C

TREATMENT OF A CERTAIN INTEGRAL

The integral

$$- \int_{\tilde{a}_2}^{\tilde{a}_1} (1/p)[(\tilde{a}_1-p)(\tilde{a}_2+p)]^{-1/2} f^{mn}(p)\, dp$$

has a singular integrand at $p=0$. In the main text it is stated that one should form the Cauchy principal value. Here we shall justify this procedure. The term arises in the evaluation of $\tilde{Q}_2^{mn}$, Eq. (67); specifically it is due to the function $I^m(p)$, defined in Eqs. (68) (70) and (71). The above expression arises from the first term on the right in Eq. (70).

According to the observations made in conjunction with Eq. (32), the occurrence of such a singularity must be expected. One deals with the evaluation of a certain double integral, and such a singularity will arise, unless one carries out the integrations in a certain sequence. In the present approach a different sequence has been applied. To give meaning to the double integral in such a case, one first excludes the point $(\tilde{x},\tilde{y})$, by a line $\tilde{\xi}=\tilde{x}\ (1-\varepsilon)$ or $\hat{\xi}=1-\varepsilon>0$. In the present context $\tilde{y}_o=0$.

The factor $1/p$ in the integrand arises from $\tilde{I}_{-2}$, Eq. (B4). Before the limits of integration are substituted the term which causes this factor reads

$$p^{-2}(1-p^2)^{-1} \frac{(q^2-p^2)^{1/2}}{q-p^2} \qquad \text{(C1)}$$

The variable of integration q is expressed by $\tilde{\xi}$ in Eq. (B2). The lower limit in $\tilde{\xi}$ in Eq. (B1), transforms into $q=1$. This appears here as the upper limit. The upper limit is $q = |p|$ for values

of p, for which that the integration over $\hat{\xi}$ ends at one of the straight lines AB or CB in Figure 14. Along the straight line BC, the upper limit is given by by $\hat{\xi}=1-\varepsilon$. Then one obtains

$$q = 1-(1-\varepsilon)-(1-p^2)$$

$$q = \varepsilon + p^2 - \varepsilon p^2.$$

By the transition from $\hat{\xi}$ to q the upper and lower limits are interchanged.

ε is assumed to be small, and obviously $p=O(\varepsilon)$ along BC. The last term on the right can, therefore, be neglected. The values of p for which this limit must be applied are found from the requirement, that at the transition value the limits from the adjacent regions are the same

$$|p| = \varepsilon + p^2$$

For ε small $\quad p^2 \ll \varepsilon$.

The transition points, therefore, lie at

$$|p| = \varepsilon.$$

Substituting these limits into the expression (C1) one obtains

$$p^{-2}(1-p^2)^{-3/2} \text{ for } p > \varepsilon$$

and $$p^2(1-p^2)^{-1}\left[\frac{1}{(1-p^2)^{1/2}} - \frac{(\varepsilon^2-p^2)^{1/2}}{\varepsilon-p^2}\right] \text{ for } p<\varepsilon$$

In the bracket we have used the fact that p and ε are small. Notice, however, that p/ε is not necessarily small. One has

$$\left[\frac{1}{(1-p^2)^{1/2}} - \frac{(\varepsilon^2-p^2)^{1/2}}{\varepsilon-p^2}\right] = \frac{1}{(1-p^2)^{1/2}(\varepsilon-p^2)}\left[\varepsilon-p^2-(\varepsilon^2-p^2)^{1/2}\right]$$

$$\approx \frac{1}{\varepsilon}\left[\frac{\varepsilon^2-2\varepsilon p^2+p^2-\varepsilon^2+p^2}{\varepsilon-p^2+(\varepsilon^2-p^2)^{1/2}}\right] = p^2 0\left(\frac{1}{\varepsilon^2}\right)$$

The critical integral then appears in the forms

$$\int_{-\tilde{a}_2}^{-\varepsilon} \frac{1}{p}\left[(\tilde{a}_1-p)(\tilde{a}_2+p)\right]^{-1/2} f^{m,n}(p)dp + \int_{-\varepsilon}^{+\varepsilon} p0\left(\frac{1}{\varepsilon^2}\right)dp$$

$$+ \int_{\varepsilon}^{\tilde{a}_1} \frac{1}{p}\left[(\tilde{a}_1-p)(\tilde{a}_2+p)\right]^{-1/2} f^{m,n}(p)dp$$

The first and third term combined are the integral interpreted as its Cauchy principal value. The integrand of the second term can be split into its symmetric and antisymmetric parts. The lowest order term of the symmetric part is

$$\text{const} \int_{-\varepsilon}^{+\varepsilon} \frac{p^2 dp}{\varepsilon^2} = 0(\varepsilon)$$

This establishes the (expected) result.

APPENDIX D

EVALUATION OF AN INTEGRAL

The expression to be evaluated is

$$I = P\int_{-a_2}^{a_1} [(a_1-p)(a_2+p)]^{-1/2} p^{-1} dp \quad , \; a_1>0, \; a_2>0$$

where P expresses that the Cauchy principal value is to be taken.

By setting

$$p = \frac{a_1-a_2}{2} + \tilde{p}\,\frac{a_1+a_2}{2}$$

one obtains
$$I = \left(\frac{a_1+a_2}{2}\right)^{-1} \tilde{I}$$

where

$$\tilde{I} = \int_{-1}^{+1} \frac{d\tilde{p}}{(1-\tilde{p}^2)^{1/2}(\tilde{p}-a)}$$

and
$$a = -\frac{a_1-a_2}{a_1+a_2}$$

Because $a_1>0$, $a_2>0$ one has

$$|a| < 1.$$

Setting
$$\tilde{p} = \frac{u^2-1}{u^2+1}$$

$$d\tilde{p} = \frac{4u\,du}{(u^2+1)^2}$$

$$(1-p^2) = \frac{4u^2}{u^2+1}$$

One obtains

$$\tilde{I} = \int_0^\infty \frac{2du}{u^2(1-a)-(1+a)} = \frac{1}{1-a}\int_0^\infty \frac{2du}{u^2-b^2} = \frac{1}{1-a}\int_{-\infty}^{+\infty} \frac{du}{u^2-b^2} \qquad \text{(D1)}$$

where $b^2 = \frac{1+a}{1-a} > 0$, since $|a|<1$.

$\tilde{I}$ is rewritten

$$\tilde{I} = \frac{1}{(1-a)}\frac{1}{2b}\int_{-\infty}^{+\infty}\left(\frac{1}{u-b} - \frac{1}{u+b}\right)du$$

Eq. (D1) shows that the path of integration can be closed at infinity, in the upper half of the complex u-plane Figure 15. The singular points $u = \pm b$ are excluded by small circles. The integral over this closed path gives zero. The residue at the two singular points are ± 1. The contributions of the small circles must be subtracted, if one forms the principal value of the integral. The sum of these contributions is zero, because the residues have opposite sign. Therefore

$$P\int_{-a_2}^{a_1} [(a_1-p)(a_2+p)]^{-1/2}p^{-1}dp = 0.$$

APPENDIX E

LEIBNITZ' RULE FOR CERTAIN INTEGRALS WITH SINGULARITIES IN THE INTEGRAND

Consider an integral

$$I(z) = \int_{x_1(z)}^{x_2(z)} f(x,z)dx$$

where the function $f(x,z)$ and the limits $x_1(z)$ and $x_2(z)$ are differentiable with respect to z. Then one obtains according to Leibnitz' rule

$$dI/dz = f(x_2(z),z)(dx_2/dz) - f(x_1(z),z)(dx_1/dz) + \int_{x_1(z)}^{x_2(z)} (\partial f(x,z)/\partial z)dx \tag{E1}$$

To derive this equation one introduces

$$x = g(u,z)$$

where the function $g(u,z)$ is chosen in such a manner that for the values of z under consideration

$$x_1(z) = g(u_1,z)$$

$$x_2(z) = g(u_2,z)$$

where u_1 and u_2 are independent of z.

We denote by $f^{(1)}$ and $f^{(2)}$ or $g^{(1)}$ and $g^{(2)}$ the derivative of f and g with respect to either the first, or second argument. For higher derivatives a corresponding notation is used, for

instance $f^{(1,1)}$ for the second derivative of f with respect to the first argument.

Then

$$I(z) = \int_{u_1}^{u_2} f(g(u,z)g^{(1)}(u,z)du$$

and

$$dI/dz = \int_{u_1}^{u_2} \{f^{(1)}(g(u,z),z)g^{(2)}(u,z)g^{(1)}(u,z)+f^{2}(g(u,z),z)g^{(1)}(uz)$$

$$+ f(g(u,z)g^{(1,2)}(u,z)\}du$$

One obtains the result Eq. (E1) by observing that in a more conventional rotation

$$f^{(1)}g^{(1)} = (\partial f/\partial x)(\partial g/\partial u)=(df/du)\Big|_{z = \text{const}}$$

Carrying out an integration by parts in the first term of the integrand one obtains

$$dI/dz = f(g(u,z),z)\ g^{(2)}(u,z)\Big|_{u_1}^{u_2} + \int_{u_1}^{u_2} \{-f(g(u(z),z)g^{(1,2)}(u,z) +$$

$$+f^{(2)}(g(u,z),z)g^{(1)}(uz) + f(g(u,z)g^{(1,2)}(u,z)\}du$$

The first and the third term of the integrand cancel.

We return to the original variable x

$$f(g(u,z),z)\Big|_{u=u_1} = f(x_1,z)$$

$$g^{(2)}(u,z)\Big|_{u=u_1} = dx_1/dz$$

and analogously for u_2. One then obtains Eq. (E1).

This approach is useful, if the function f has a singularity at one of the limits, and if one has to form higher derivatives. Consider a simple example:

$$I(z) = \int_0^z (z-x)^{1/2}dx \tag{E2}$$

The result is obvious

$$I(z) = -(2/3)(z-x)^{3/2}\Big|_{x=0}^{z} = (2/3)z^{3/2} \tag{E3}$$

$$\frac{dI}{dz} = z^{1/2} \;;\; d^2I/dz^2 = (1/2)z^{-1/2} \qquad \text{(E4),(E5)}$$

Let us forego the direct integration, and carry out the differentiation in Eq. (E2) with respect to z according to Leibnitz' rule. The integrand in Eq. (E2) vanishes at the upper limit z.

Accordingly,

$$\frac{dI}{dz} = (1/2)\int_0^z (z-x)^{-1/2}dx = -(z-x)^{1/2}\Big|_0^z = z^{1/2}$$

Although the integrand in Eq. (E2) has a square root singularity at the upper limit, one obtains the correct result, Eq. (E4). The procedure fails for the second derivative

$$\frac{d^2I}{dz^2} = (1/2)\ (z-x)^{-1/2} \int_0^z -(1/4) \Big|_0^z (z-x)^{-3/2}dx$$

At the upper limit the first term becomes infinite. It is true the integral becomes infinite, too, and the two "infinities" will cancel, but this observation still does not give a result.

Proceeding in the manner shown above we introduce

$$(x-z) = u$$

Then

$$I(z) = \int_0^z u^{1/2}du$$

Because the integrand has no singularity at the upper limit, Leibnitz' rule can be applied

$$dI/dz = u^{1/2} \Big|_0^z = z^{1/2}$$

The second derivative is then trivial.

In the context of this report, the problem appears in a more complicated form.

$$I(z^2) = \int_{a(z^2)}^{b} (x-a(z^2))^{1/2}f(x,z^2)dx$$

where the derivatives of f and a with respect to the two variables exist. We set

$$s = z^2$$

Then

$$I(s) = \int_{a(s)}^{b} (x-a(s))^{1/2} f(x,s)dx$$

The integrand vanishes at the lower limit. Application of Leibnitz' rule gives the correct result.

$$dI/ds = \int_{a(s)}^{b} [-(1/2)(x-a(s))^{-1/2} a'(s)f(x,s)+(x-(a(s))^{1/2} f^{(2)}(x,s)]dx \tag{E5}$$

The procedure fails for the second derivative. Therefore, one sets

$$x - a(s) = u$$

Then

$$I(s) = \int_{u=0}^{u=b-a(s)} u^{1/2} f(a(s)+u,s)du$$

The integrand is regular at the upper (s-dependent) limit. Therefore Leibnitz' rule can be applied

$$(dI/ds) = -u^{1/2} f(a(s)+u,s)a'(s)\Big|_{w=b-a(s)} \tag{E6}$$

$$+ \int_{u=0}^{b-a(s)} u^{1/2}[f^{(1)}(a(s)+u,s)a'(s) + f^{(2)}(a(s)+u,s]du$$

To obtain Eq. (E5) an integration by parts must be carried out. Notice that

$$\int f^{(1)}(a(s)+u,s)du = f(a(s)+u,s)$$

An integration by parts of the first term in the integrand gives outside of the integral a term which cancels the first term on the right in Eq. (E6). Therefore

$$\frac{dI}{ds} = \int_{u=0}^{b-a(s)} [(1/2)u^{-1/2}f(a(s)+u,s)a'(s) + u^{1/2}f^{(2)}(a(s)+a,s)]du$$

Eq. (E5) is obtained after one reintroduces x. The second derivative (d^2I/ds^2) from Eq. (E6) can be found without encountering an infinity.

In this report, one needs derivatives with respect to z but only for $z=0$. One has

$$(dI/dz) = 2z(dI/du)$$

$$(d^2I/dz^2) = 2(dI/du) + 4z^2(d^2I/du^2)$$

Hence

$$dI/dz\Big|_{z=0} = 0 \tag{E7}$$

$$d^2I/dz^2\Big|_{z=0} = 2(dI/du)$$

$$= \int_{a(z^2)}^{b} -(x-a(z^2)^{-1/2}(da/dz^2)f(x,(z)^2)+2(x-a(t^2)^{1/2}f^{(2)}(x,z^2)dx\Big|_{z=0}$$

$$= 2\int_{a(z^2)}^{b} \frac{d}{d(z^2)} [(x-(a(z^2)^{1/2}f(x,z^2)]dx\Big|_{z=0} \tag{E8}$$

The results Eqs. (E7) and (E8) could have been anticipated on intuitive grounds.

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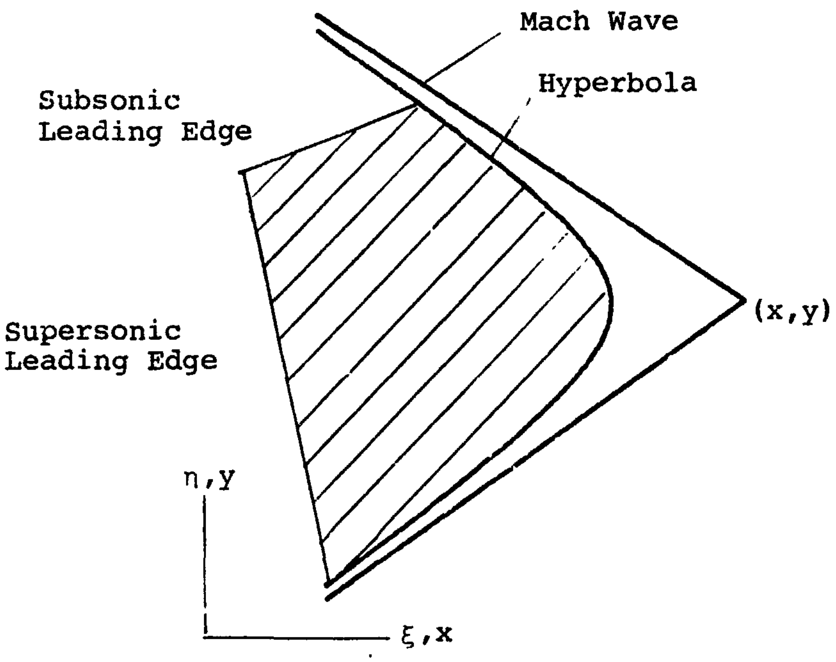

Figure 1. Region of Integration with Subsonic and Supersonic Leading Edges.

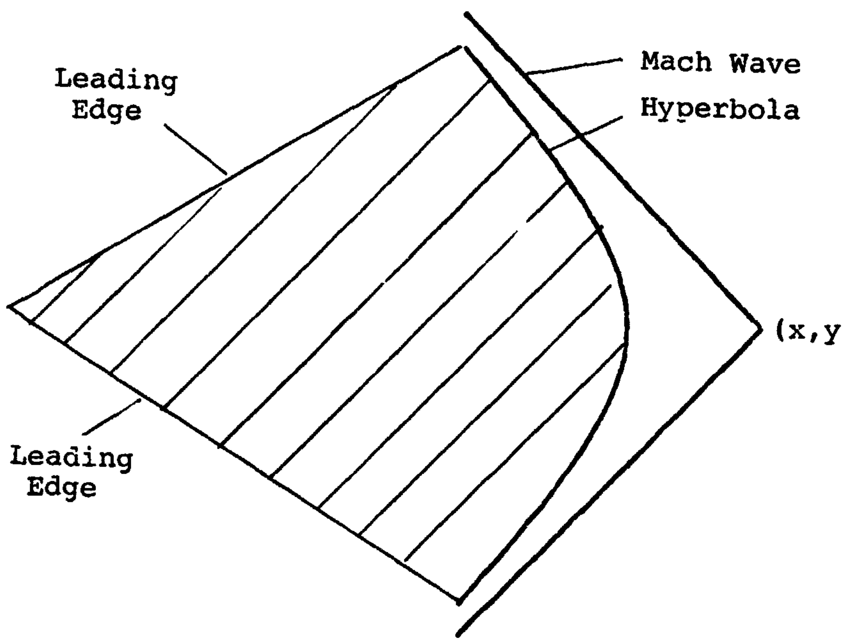

Figure 2. Region of Integration with Subsonic Leading Edges.

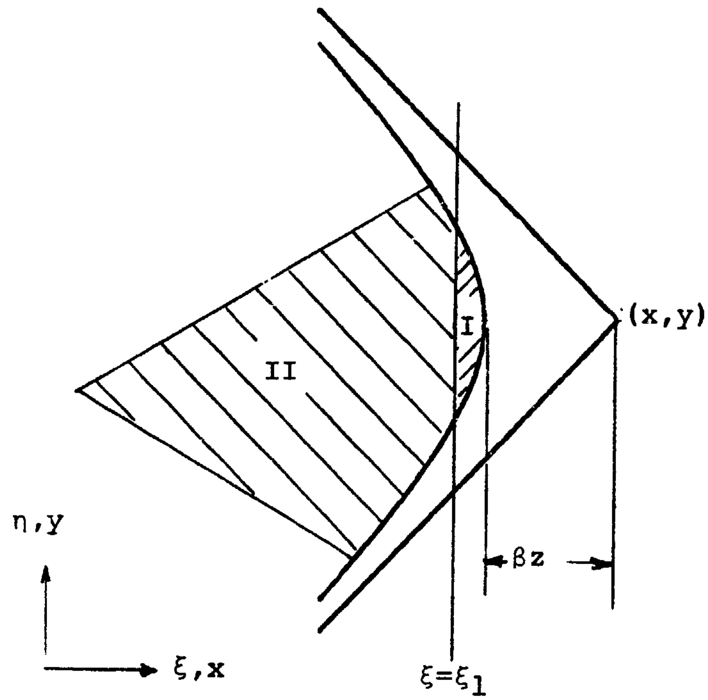

Figure 3. Subdivision of the Region of Integration.

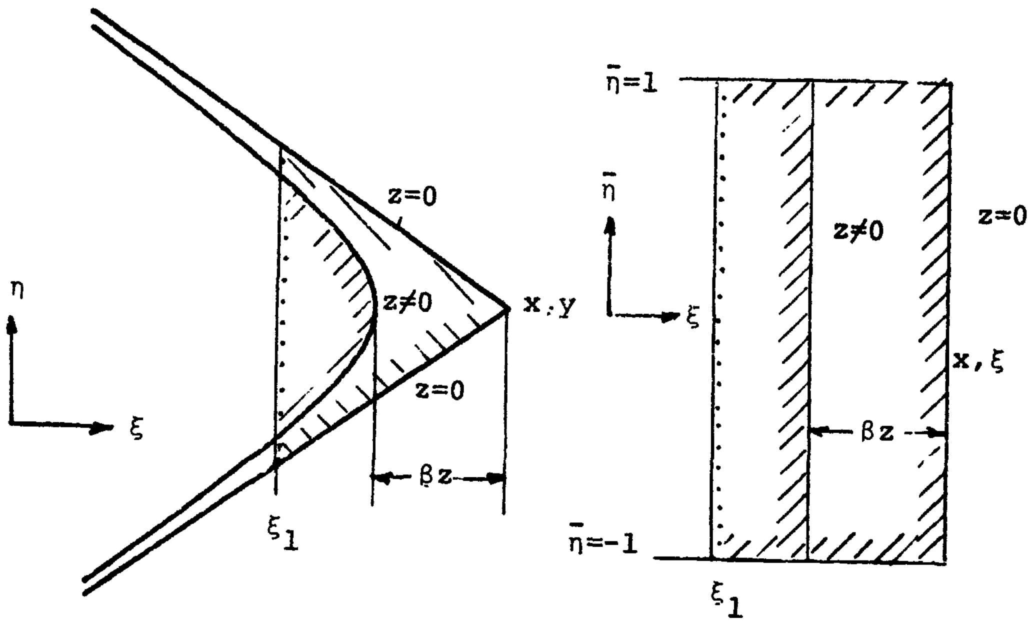

Figure 4. Region of Integration I for z=0 and z≠0.

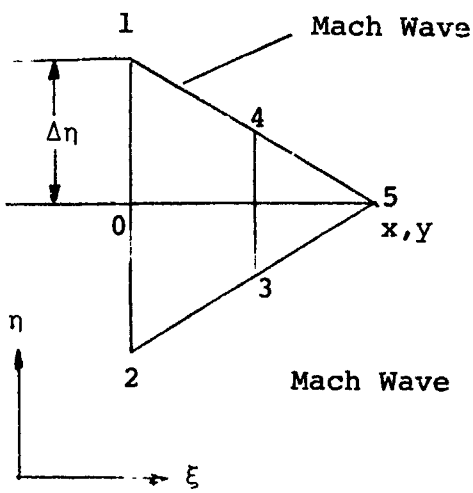

Figure 5. Evaluation of the Double Integral in Eq. (32) in the Vicinity of a Control Point.

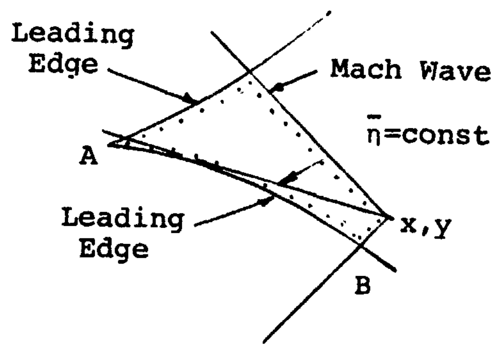

Figure 6. Region of Integration for z=0 with Line $\bar{\eta}$ = const Tangent to one of the Leading Edges.

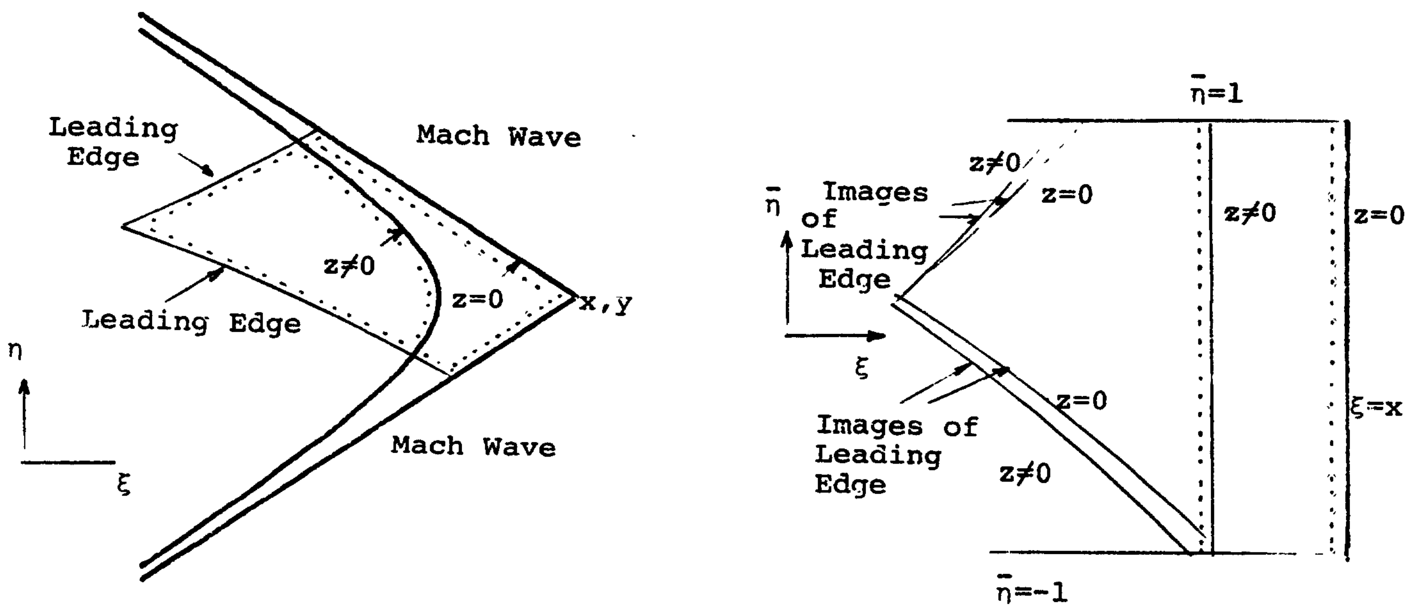

Figure 7. Region of Integration in the $\xi\eta$ and $\xi\bar{\eta}$ Coordinates for $z=0$ and $z \neq 0$.

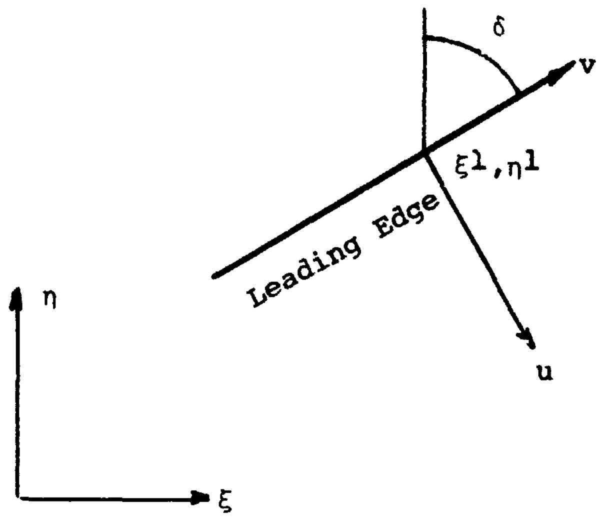

Figure 8. Local Coordinates at a Point of the Leading Edge.

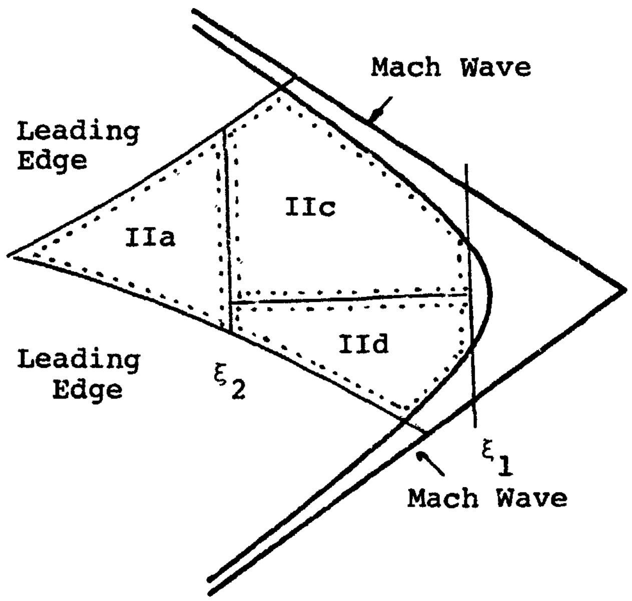

Figure 9. Subdivision of the Region of Integration II.

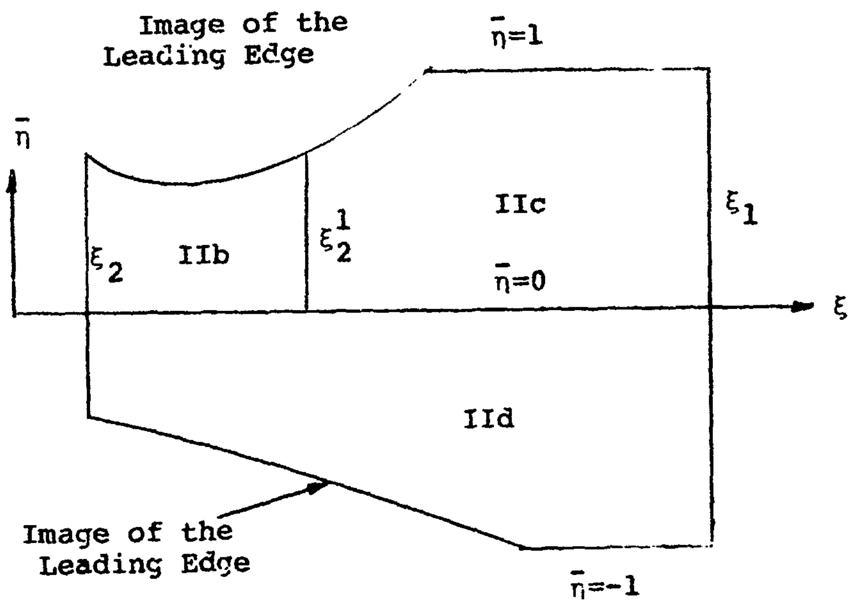

Figure 10. Subdivision of the Region II if the Image of the Leading Edge has a Point where it is Tangent to a Line $\bar{\eta}$ = const.

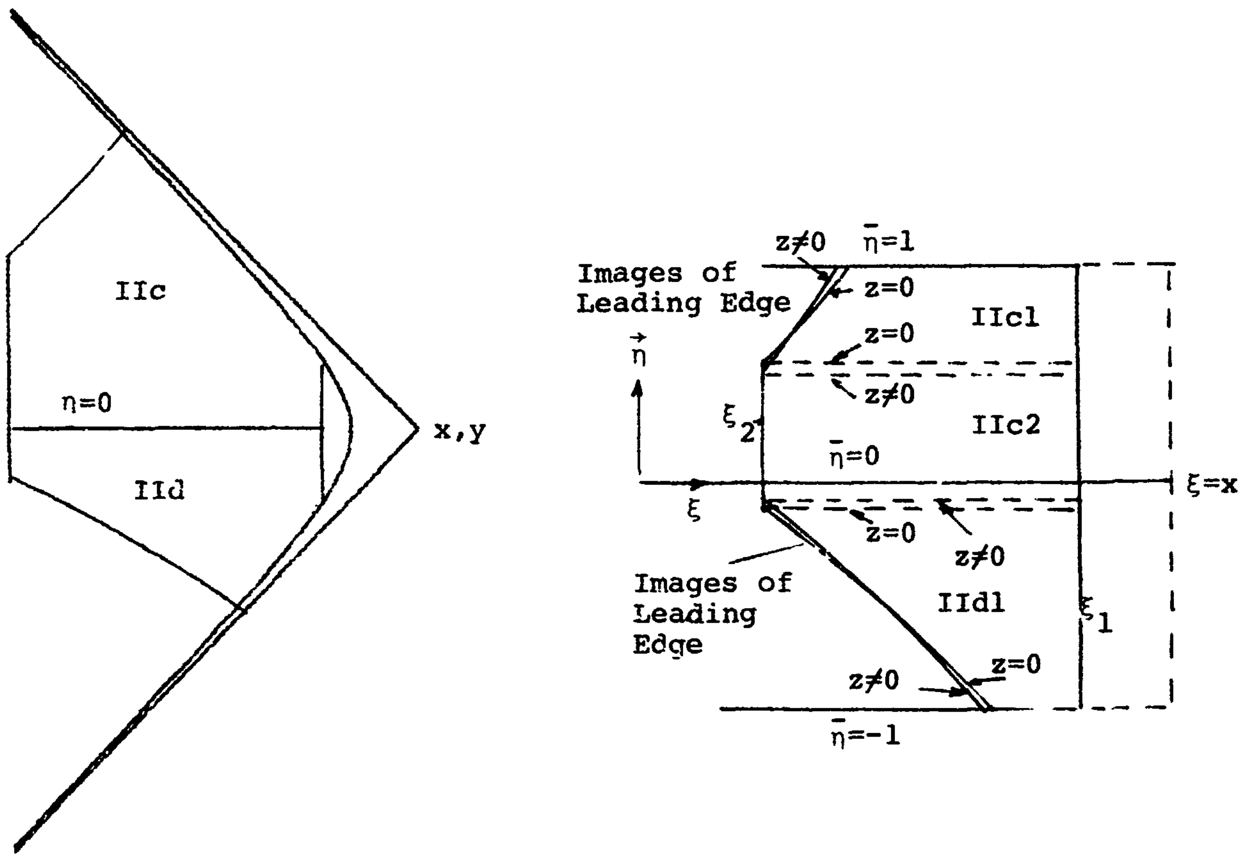

Figure 11. Subdivision of Region II in the $\xi\eta$ Plane with Further Subdivisions in the $\xi\bar{\eta}$ Plane.

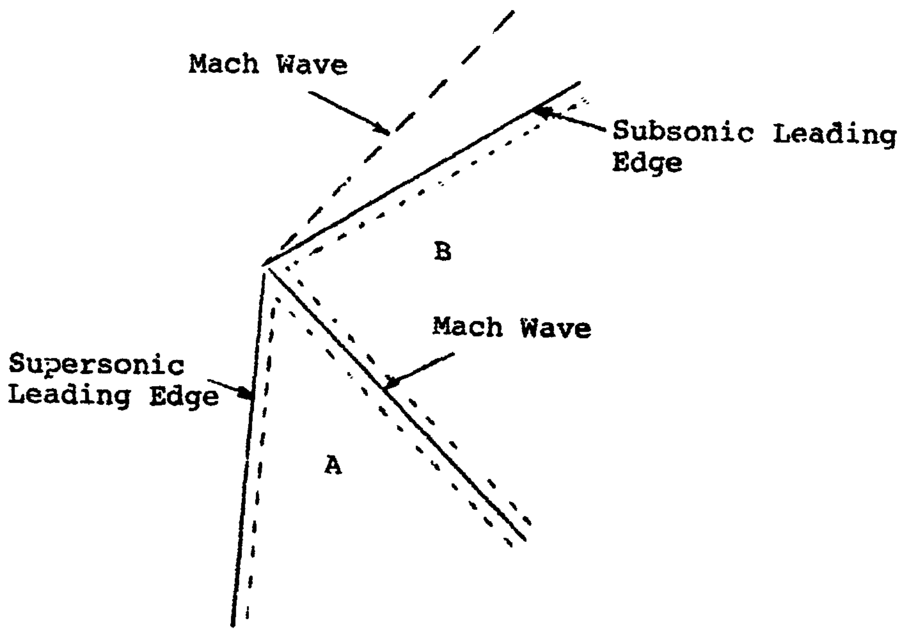

Figure 12. Regions A and B in a Problem with a Subsonic and a Supersonic Leading Edge.

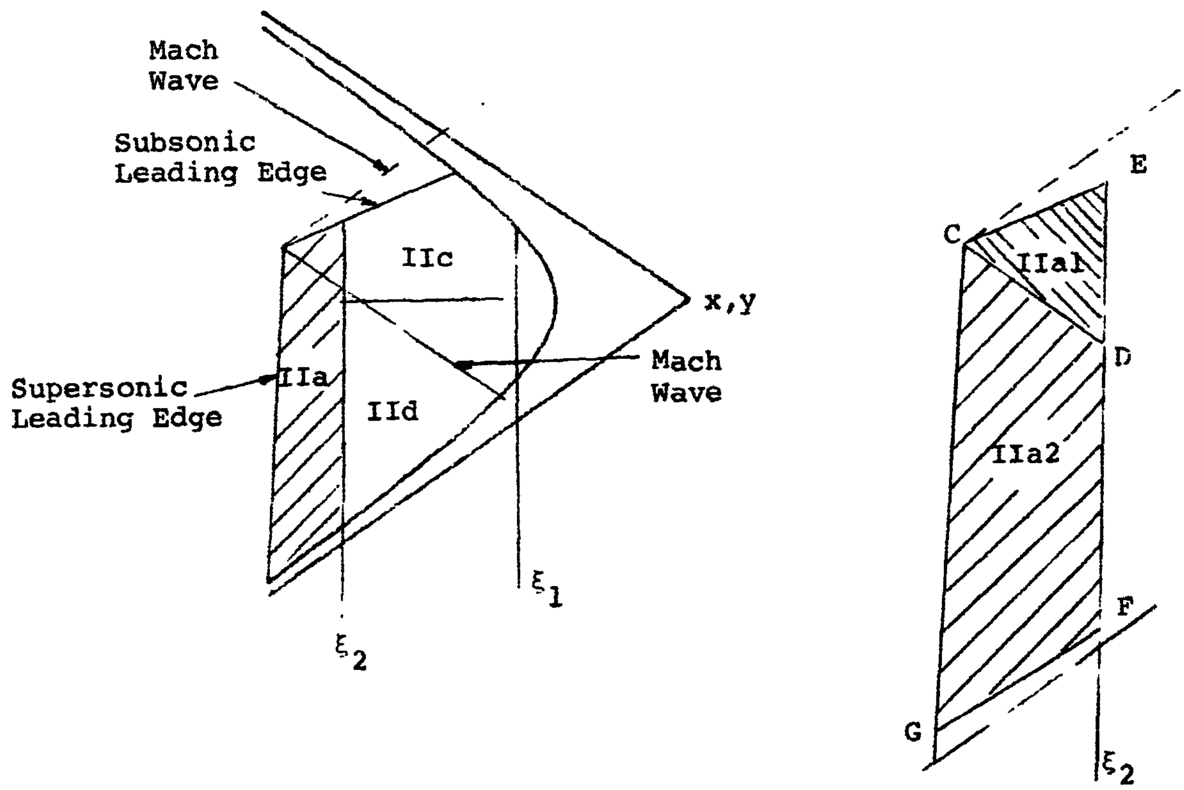

Figure 13. Problem with a Subsonic and a Supersonic Leading Edge, Region IIa and its Subregions IIa1 and IIa2.

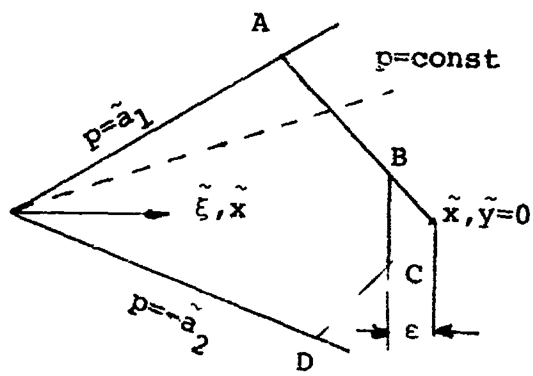

Figure 14. Truncation of the Area of Integration by the Straight Line BC ($\xi = x - \varepsilon$)

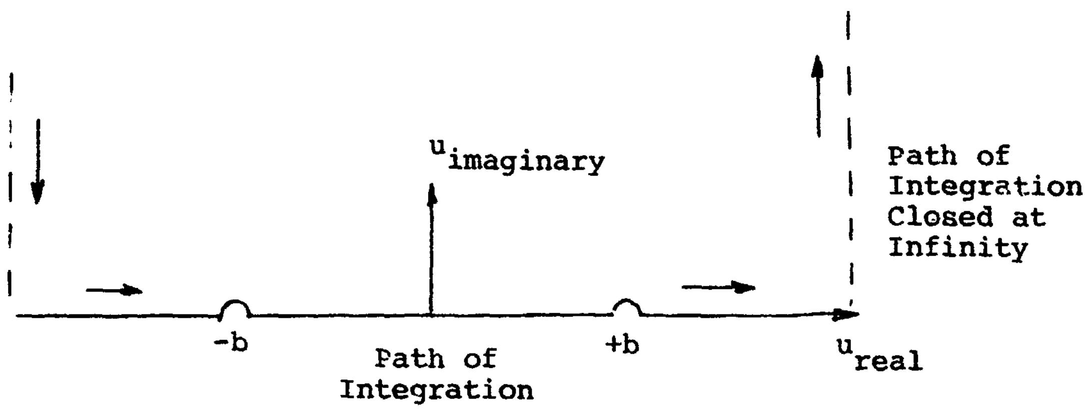

Figure 15. Evaluation of a Certain Integral in the Complex u-Plane.

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